NEW AGE

APPLIED NONLINEAR PROGRAMMING

Sanjay Sharma

NEW AGE INTERNATIONAL PUBLISHERS

APPLIED
NONLINEAR
PROGRAMMING

THIS PAGE IS BLANK

APPLIED NONLINEAR PROGRAMMING

Sanjay Sharma

PUBLISHING FOR ONE WORLD

NEW AGE INTERNATIONAL (P) LIMITED, PUBLISHERS

New Delhi • Bangalore • Chennai • Cochin • Guwahati • Hyderabad
Jalandhar • Kolkata • Lucknow • Mumbai • Ranchi

Visit us at **www.newagepublishers.com**

Copyright © 2006, New Age International (P) Ltd., Publishers
Published by New Age International (P) Ltd., Publishers

All rights reserved.

No part of this ebook may be reproduced in any form, by photostat, microfilm, xerography, or any other means, or incorporated into any information retrieval system, electronic or mechanical, without the written permission of the publisher. *All inquiries should be emailed to* ***rights@newagepublishers.com***

ISBN (10) : 81-224-2339-6

ISBN (13) : 978-81-224-2339-6

PUBLISHING FOR ONE WORLD

NEW AGE INTERNATIONAL (P) LIMITED, PUBLISHERS
4835/24, Ansari Road, Daryaganj, New Delhi - 110002
Visit us at **www.newagepublishers.com**

PREFACE

In most of the practical situations, the objective function and/or constraints are nonlinear. The solution methods to deal with such problems may be called as nonlinear programming. The present book is focussed on applied nonlinear programming.

A general introduction is provided discussing various industrial/managerial applications. Convex and concave functions are explained for single variaSble and multi-variable examples. As a special case of nonlinear programming, geometric programming is also discussed. It is expected that the book will be a useful reference/text for professionals/students of engineering and management disciplines including operations research.

Dr. SANJAY SHARMA

THIS PAGE IS BLANK

CONTENTS

THIS PAGE IS BLANK

1

INTRODUCTION

Any optimization problem essentially consists of an objective function. Depending upon the nature of objective function (O.F.), there is a need to either maximize or minimize it. For example,

(*a*) Maximize the profit

(*b*) Maximize the reliability of an equipment

(*c*) Minimize the cost

(*d*) Minimize the weight of an engineering component or structure, etc.

If certain constraints are imposed, then it is referred to as constrained optimization problem. In the absence of any constraint, it is an unconstrained problem.

Linear programming (LP) methods are useful for the situation when O.F. as well as constraints are the linear functions. Such problems can be solved using the simplex algorithm. Nonlinear programming (NLP) is referred to the followings:

(*a*) Nonlinear O.F. and linear constraints

(*b*) Nonlinear O.F. and nonlinear constraints

(*c*) Unconstrained nonlinear O.F.

1.1 Applications

In the industrial/business scenario, there are numerous applications of NLP. Some of these are as follows:

(a) Procurement

The industries procure the raw materials or input items/components regularly. These are frequently procured in suitable lot sizes. Relevant total annual cost is the sum of ordering cost and inventory holding cost. If the lot size is large, then there are less number of orders in one year and thus the annual ordering cost is less. But at the same time, the inventory holding cost is increased. Considering the constant demand rate, the total cost function is non-linear and as shown in Fig. 1.1.

There is an appropriate lot size corresponding to which the total annual cost is at minimum level. After formulating the nonlinear total cost function in terms of the lot size, it is optimized in order to evaluate the desired procurement quantity. This quantity may be procured periodically as soon as the stock approaches at zero level.

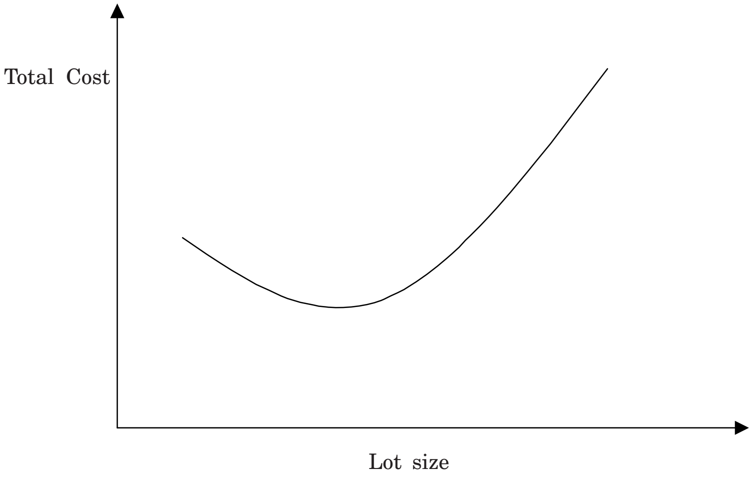


Fig. 1.1: Total cost function concerning procurement.

(b) Manufacturing

If a machine or facility is setup for a particular kind of product, then it is ready to produce that item. The setup cost may include salaries/wages of engineers/workers for the time period during which they are engaged while the machine is being setup. In addition to this, the cost of trial run etc., if any, may be taken into consideration. Any number of items may be manufactured in one setup before it is changed for another variety of product. There are fewer number of setups in a year if produced quantity per unit setup is more. Large number of setups are needed in case of less number of units manufactured and accordingly annual setup cost will be high. This relationship is shown in Fig. 1.2.

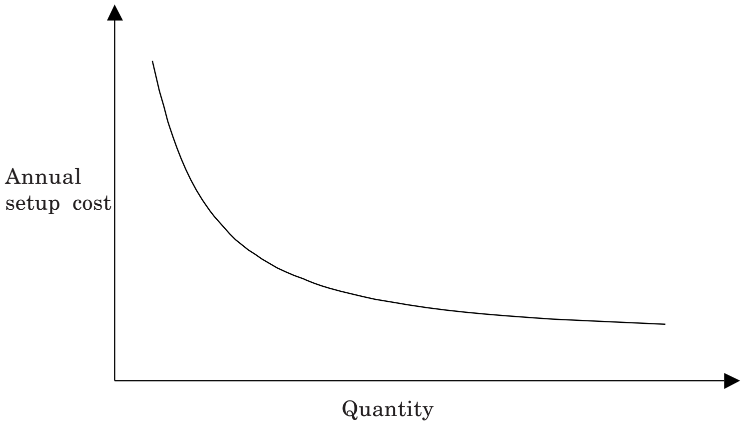


Fig. 1.2: Variation of setup cost with quantity.

Large number of units may be manufactured in a setup if only this cost is considered. But at the same time, there will be large number of items in the inventory for a certain time period and inventory holding cost will be high. Inventory holding cost per year is related to quantity as shown in Fig. 1.3.

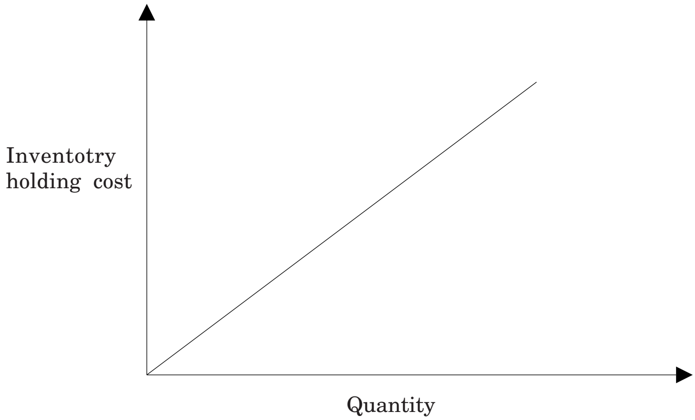


Fig. 1.3: Variation of holding cost with quantity.

As the total annual cost is the sum of setup cost and holding cost, combined effect of Fig. 1.2 and Fig. 1.3 will yield the total cost which is similar to Fig. 1.1. Therefore in the present case, total relevant cost is a nonlinear function. Analysis of this cost function is concerned with the trade-off between machine setup and carrying cost. Objective is to evaluate an optimum manufacturing quantity or production lot size so that the total cost is minimum.

(c) Break-even analysis

For the successful operation of any industry, it is of interest to know the production level at which there is no profit-no loss. This is known as break-even point. If the manufactured and sold quantity is less than this point, there are losses. Profits are earned if produced and marketed quantity is more than the break-even point. Break-even analysis is the interaction of sales revenue and total cost where the total cost is the sum of fixed and variable cost.

Fixed cost is concerned with the investments made in capital assets such as machinery and plant. Variable cost is concerned with the actual production cost and is proportional to the quantity. Total cost line is shown in Fig. 1.4 along with sales revenue. Sales revenue is the multiplication of sold quantity and sales price per unit.

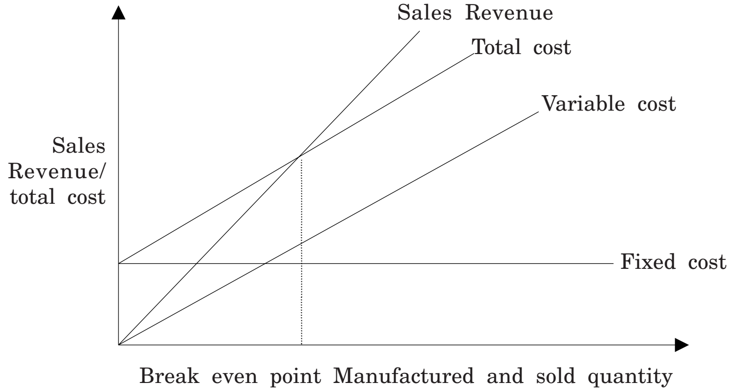


Fig. 1.4: Interaction of linear total cost and sales revenue.

Total cost is shown to be linear in Fig. 1.4. However, in practice, depending on the nature of variable cost and other factors, the total cost may be non-linear as shown in Fig. 1.5.

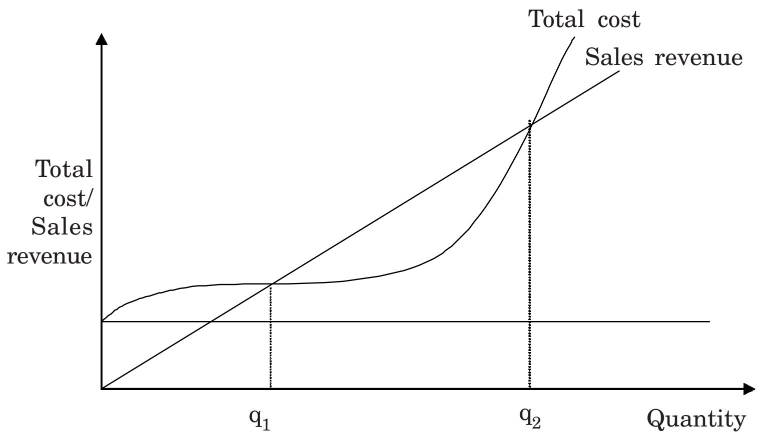


Fig. 1.5: Interaction between nonlinear total cost and sales revenue.

Nonlinear total cost function and sales revenue line intersect at two points corresponding to quantity q_1 and q_2. Therefore two break-even points exist in the visible range of quantity. As the profit = sales revenue – total cost, it is zero if either quantity q_1 or q_2 are produced and sold. There is certain quantity which is more than q_1 and less than q_2, at which maximum profit can be achieved.

In more complex situations, both the total cost and sales revenue functions may be nonlinear.

(d) Logistics

Logistics is associated with the timely delivery of goods etc. at desired places. A finished product requires raw materials, purchased and in-house fabricated components and different input items. All of them are needed at certain specified time and therefore transportation of items becomes a significant issue. Transportation cost needs to be included in the models explicitly. Similarly effective shipment of finished items are important for the customer satisfaction with the overall objective of total cost minimization. Incoming of the input items and outgoing of the finished items are also shown in Fig. 1.6.

Logistic support plays an important role in the supply chain management (SCM). Emphasis of the SCM is on integrating several activities including procurement, manufacturing, dispatch of end items to warehouses/dealers, transportation etc. In order to consider production and purchase of the input items together, frequency of ordering in a manufacturing cycle may be included in the total cost formulation. Similarly despatch of the finished component periodically to the destination firm in smaller lots, may be modeled.

Temporary price discounts are frequently announced by the business firms. An organization takes the advantage of this and purchases larger quantities during the period for which discount was effective. Potential cost benefit is maximized. In another situation, an increase in the price of an item may be declared well in advance. Total relevant cost may be reduced by procuring larger quantities before the price increase becomes effective. Potential cost benefit is formulated and the objective is to maximize this function.

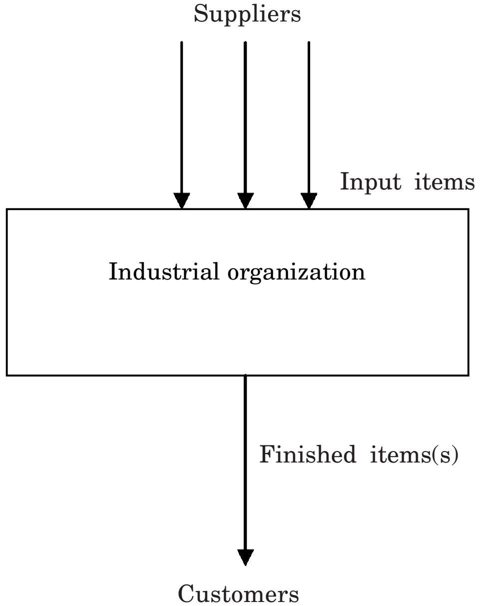


Fig. 1.6: Incoming and outgoing of the items.

The present section discusses some real life applications in which a nonlinear objective function is formulated. These functions are either maximized or minimized depending on the case. Maxima and minima are explained next.

1.2 MAXIMA AND MINIMA

Consider any function $f(x)$ as shown in Fig. 1.7. $f(x)$ is defined in the range of x, [A, B]. $f(x)$ is having its maximum value at x^* which is optimal.

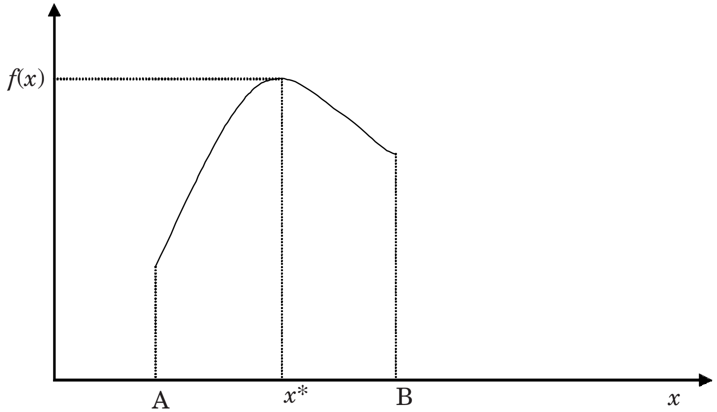


Fig. 1.7: A function $f(x)$.

Refer to Fig. 1.8 in which behavior of a function $f(x)$ with respect to x is represented in the range [A, B] of x. Three number of maximum points are visible *i.e.* 1, 2, and 3. these are called as local or relative maxima.

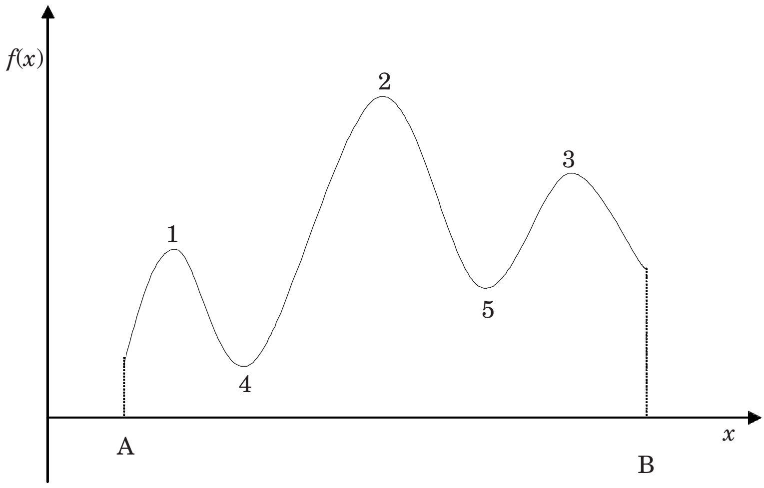


Fig. 1.8: Local and global maxima/minima.

Among the values of x at point 1, 2 and 3, optimal solution with respect to maximization of $f(x)$, lies at point 2. This is because the value of $f(x)$ at point 2 is maximum as compared to that at point1 and 3. Therefore point 2 is referred to as global maximum.

Similarly, in view of the minimization, point 4 and 5 are referred to as local or relative minima. Point 5 is called as global minimum because value of $f(x)$ at this point is the least.

Local maximum or local minimum can also be called a local extremum.

For a maximum or minimum,

$$\frac{df(x)}{dx} = 0 \qquad ...(1.1)$$

For the optimality, second order derivative should be negative in case of maxima and a positive in case of minima, *i.e.*

$$\frac{d^2f(x)}{dx^2} < 0, \text{ for maxima} \qquad ...(1.2)$$

and

$$\frac{d^2f(x)}{dx^2} > 0, \text{ for minima} \qquad ...(1.3)$$

1.3 CONVEX AND CONCAVE FUNCTION

Consider a linear function,

$$y = f(x) = a + bx \qquad ...(1.4)$$

Assume two end points x_1 and x_2 on the x-axis as shown in Fig. 1.9. x-coordinate for any intermediate point on the straight line can be represented by, $\alpha x_1 + (1 - \alpha) x_2$ for á lying between 0 and 1. The end points x_1 and x_2 are obtained by substituting $\alpha = 1$ and 0 respectively. To generalize, coordinate $-x$ is,

$$\alpha x_1 + (1 - \alpha) x_2, \text{ for } 0 \leq \alpha \leq 1$$

Substituting this value of x in equation (1.4), coordinate $-y$ is,

$$y = f(x) = a + b [\alpha x_1 + (1 - \alpha) x_2]$$

$$= a + b x_2 + a \alpha + b \alpha x_1 - a \alpha - b \alpha x_2$$

(adding and subtracting a α)

$$= (a + bx_2) + \alpha(a + bx_1) - \alpha(a + bx_2)$$
$$= \alpha\,(a + bx_1) + (1 - \alpha)\,(a + bx_2)$$
$$= \alpha f(x_1) + (1 - \alpha)\,f(x_2) \qquad ...(1.5)$$

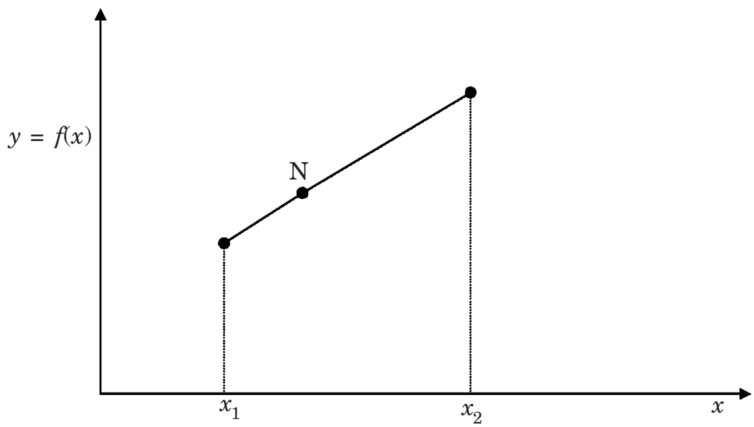


Fig. 1.9: A linear function.

Coordinates of any point, say N on the straight line (Fig. 1.9), can be written as,

$$[\alpha\, x_1 + (1 - \alpha)\, x_2, \quad \alpha\, f(x_1) + (1 - \alpha)\, f(x_2)]$$

This discussion with reference to a linear function is helpful in understanding the convex and concave functions.

1.3.1 Convex Function

A nonlinear function $f(x)$ is shown in Fig. 1.10. This function is of such a nature that the line joining any two selected points on this function will never be below this function. In other words, viewing from the bottom, this function or curve will look convex.

Select any two points on the convex function, say P and Q. From any point on the line PQ *i.e.* N, draw a vertical line NM which meets the convex function $f(x)$ at point L.

As discussed before, x-coordinate of point M is,

$$\alpha\, x_1 + (1 - \alpha)\, x_2,\ 0 \le \alpha \le 1$$

Value of $f(x)$ at point L can be written as,

$$f[\alpha x_1 + (1 - \alpha)\, x_2]$$

Value of $f(x)$ at point N on the straight line is given as, $\alpha\, f(x_1) + (1 - \alpha)\, f(x_2)$, using equation (1.5)

From Fig. 1.10, it can be observed that for a convex function,

$$\alpha f(x_1) + (1 - \alpha) f(x_2) \geq f[\alpha x_1 + (1 - \alpha) x_2],\ 0 \leq \alpha \leq 1 \quad \text{...(1.6)}$$

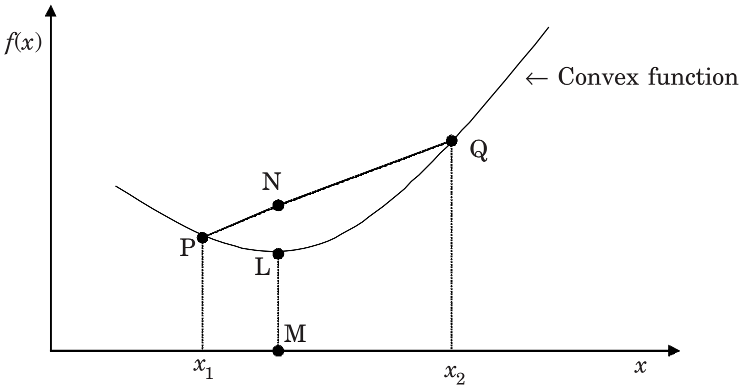


Fig. 1.10: Convex function.

1.3.2 Concave Function

Fig. 1.11 represents a nonlinear function $f(x)$. Select any two points, say, P and Q. The line joining P and Q (or any other selected points on the function) will never be above this function $f(x)$. Such a function is known as concave function. This will look concave if we view it from bottom side.

Locate any point L on the function above the line PQ and draw a vertical line LM which intersects line PQ at point N. Obviously x-coordinate for point L and N are similar *i.e.*

$\alpha x_1 + (1 - \alpha) x_2$, as discussed before.

From (1.5), y-coordinate for point N = $\alpha f(x_1) + (1 - \alpha) f(x_2)$, and

y-coordinate for point L = $f[\alpha x_1 + (1 - \alpha) x_2]$.

Using the property of a concave function,

$$\alpha f(x_1) + (1 - \alpha) f(x_2) \leq f[\alpha x_1 + (1 - \alpha) x_2],\ 0 \leq \alpha \leq 1 \quad \text{...(1.7)}$$

Differences between convex and concave functions are summarized in Table 1.1.

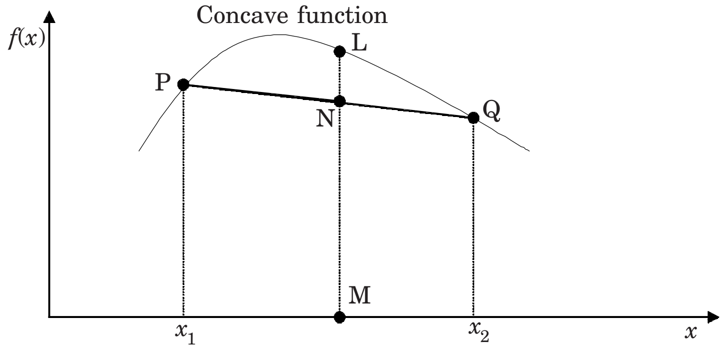


Fig. 1.11: Concave function

Table 1.1: Difference between convex and concave functions

	Convex function	*Concave function*
(1)	Line joining any two above will never be below this function.	Line joining any two points will never be above this function.
(2)	$\alpha f(x_1) + (1 - \alpha) f(x_2) \geq f[\alpha x_1 + (1 - \alpha) x_2], 0 \leq \alpha \leq 1$	$\alpha f(x_1) + (1 - \alpha) f(x_2) \leq f[\alpha x_1 + (1 - \alpha) x_2], 0 \leq \alpha \leq 1$
(3)	Non-negative second order derivative indicates a function to be convex.	Non-positive second order derivative indicates a function to be concave.
(4)	In order to minimize a convex function, $\frac{d}{dx} f(x) = 0$ yields the optimal solution.	In order to maximize a concave function, $\frac{d}{dx} f(x) = 0$ yields the optimal solution.
(5)	If $-f(x)$ is concave, then $f(x)$ is a convex function.	If $-f(x)$ is convex, then $f(x)$ is a concave function.

Example 1. *Test whether* $f(x) = \frac{18 \times 10^4}{x} + \frac{x}{2}$, *is a convex function for positive values of x. Also find out the optimal solution by minimizing it.*

Solution.

Second order derivative, $\frac{d^2 f(x)}{dx^2} = \frac{36 \times 10^4}{x^3}$

As this is positive, the function $f(x)$ is convex.

An optimal solution is obtained by differentiating $f(x)$ with respect to x and equating it to zero.

$$\frac{df(x)}{dx} = \frac{18 \times 10^4}{x^2} + \frac{1}{2} = 0$$

or optimal value of $x = 600$

Substituting $x^* = 600$ in $f(x)$, optimal value of

$$f(x) = \frac{18 \times 10^4}{600} + \frac{600}{2}$$

$$= 600$$

Example 1.2. *Show that f(x) = – (45000/x) – 2x, is a concave function for positive values of x and also obtain the optimal value of x.*

Solution.

$$\frac{d^2f(x)}{dx^2} = \frac{-90000}{x^3}$$

As this is negative, $f(x)$ is a concave function. In order to maximize this function.

$$\frac{df(x)}{dx} = 0$$

or $$\frac{45000}{x^2} - 2 = 0$$

or $$x^* = 150$$

1.4 CLASSIFICATION

In order to optimize nonlinear objective function, several methods are used in practice. The present classification may not be exhaustive. However, depending on various considerations, these methods are classified as flows:-

(a) As per the number of variables

(*i*) Single variable optimization

(*ii*) Multivariable optimization-if more than one variable are present in the O.F.

(b) Depending on the search procedure

Search for the optimum value of any variable will start from a suitable initial point. After certain number of iterations, it is expected that the goal will be achieved.

Few methods are as follows:

(*i*) *Unrestricted search*–When no idea is available for the range in which any optimum variable may lie, then a search is made without any restrictions.

(*ii*) *Restricted search.*

(*iii*) *Method of golden section*–Initial interval is known in which a variable lies and a unimodal function is optimized.

(*iv*) *Quadratic interpolation*–If any function can be approximated by the quadratic function, then a minimum value of x is obtained using, $f(x) = a\,x^2 + bx + c$.

Later this minimum value is substituted in actual function and an iterative process is continued to achieve any desirable accuracy.

(*v*) Numerical methods.

(c) Depending on whether the constraints are present

If no constraints are imposed on the O.F., then it is referred to as an unconstrained optimization. Else, the NLP problem may be as follows:

Optimize a nonlinear function,

Subject to certain constraints where these constraints may be in one or more of the forms,

$$c(x) \le 0$$

$$d(x) \ge 0$$

$$e(x) = 0$$

$c(x)$, $d(x)$ and $e(x)$ are functions in terms of decision variables and represent the constraints. First two types of constraints are in the inequality form. Third type of constraint i.e., $e(x) = 0$, is an equality form.

(d) Specific case of nonlinear O.F.

Suppose that the O.F. is the sum of certain number of components where each component is like,

$$c_i \, x_1^{a_{1i}} \cdot x_2^{a_{2i}} \cdot x_3^{a_{3i}} \ldots x_n^{a_{ni}}$$

c_i = Positive coefficient

$a_{1i}, a_{2i}, \ldots a_{ni}$ = Real exponents

$x_1, x_2, \ldots x_n$ = Positive variables.

$$\text{Now, O.F., } f(x) = \sum_{i=1}^{N} c_i \cdot x_1^{a_{1i}} \cdot x_2^{a_{2i}} \ldots x_n^{a_{ni}}$$

where N = Number of components in the O.F.

n = Number of variables in the problem.

Such a function is known as a posynomial. In order to minimize posynomial functions, geometric programming is effectively used.

(e) Integrality requirement for the variables

Usually the solutions obtained for an optimization problem give values in fractions such as 14.3, 201.57 etc. While in many real life applications, optimum variables need to be evaluated in terms of exact integers. Examples may be,

(*i*) How many optimum number of employees are required?

(*ii*) Number of components needed to be manufactured, which will be used later in the assembly of any finished product.

(*iii*) Number of cycles for procurement of input items or raw materials in the context of supply chain management.

(*iv*) Optimum number of production cycles in a year so that the total minimum cost can be achieved.

Nonlinear integer programming problems may be categorized as follows:

(1) **All integer programming problems–**This refers to the cases where all the design variables are needed to be integers. This is also called pure integer programming problem.

(2) **Mixed integer programming problems–**This refers to the cases in which there is no need to obtain the integer optimum of all variables. Rather integrality

requirement is justified for few variables only. In other words, some of the variables in a set, may have fractional values, whereas remaining must have integer values.

The classification as discussed above, is also summarized briefly in Fig. 1.12. Some of the methods may also be used satisfactorily for other categories. In addition to that, heuristic search procedures may also be developed depending on the analysis of various constraints imposed on the O.F.

NLP is introduced in the present chapter. One variable optimization is discussed next.

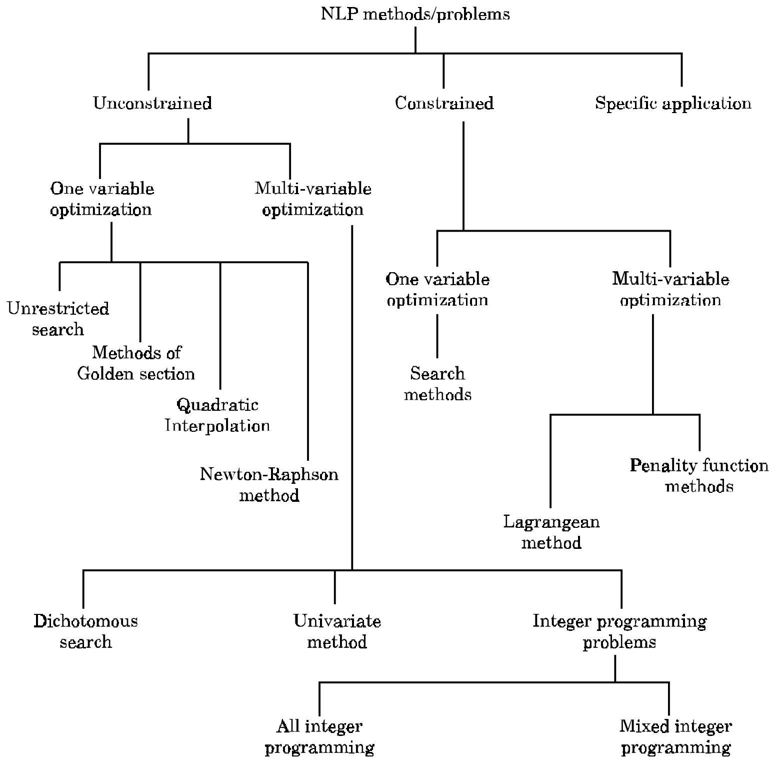


Fig. 1.12: Brief classification of NLP problems/methods.

2

ONE VARIABLE OPTIMIZATION

If any function has only one variable, then its maximization or minimization will be referred to one variable or single variable optimization. For example,

Maximize $4x - 7x^2$,

Minimize $8x^2 - 3x$

These are unconstrained one variable optimization problems. If certain constraints are imposed, such as,

(*a*) Maximize $4x - 7x^2$

subject to $x \geq 0.3$

(*b*) Minimize $8x^2 - 3x$

subject to $x \leq 0.15$

Then these will become constrained one variable optimization problems. Constraints may be imposed after obtaining the optimum of a function in an unconstrained form.

Several methods are available for optimization of one variable problem. As discussed in section 1.3, the function is differentiated with respect to a variable and equated to zero. Further some of the methods are also explained in the present chapter

1. Unrestricted search
2. Method of golden section
3. Quadratic interpolation
4. Newton-Raphson method

2.1 UNRESTRICTED SEARCH

When there is no idea of the range in which an optimum variable may lie, the search for the optimum is unrestricted. A suitable initial point is needed in order to begin the search procedure. As shown in Fig. 2.1, an optimum is shown by sign X. Assume initial point as x = 0. 0 from where a search is to be started. It is like finding an address in an unfamiliar city.

From the initial point i.e. x = 0.0, a decision is to be made whether to move in positive or negative direction. Value of x is increased or decreased in suitable step length till a close range of x, in which an optimum may lie, is not obtained. An exact optimum is achieved in this close range using smaller step length.

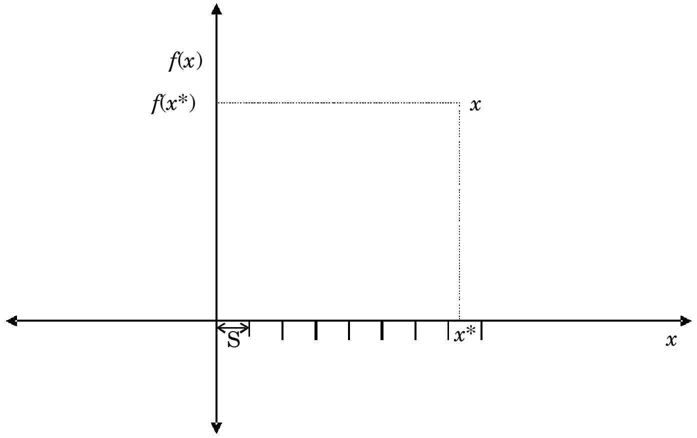


Fig. 2.1: Searching for an optimum with step length s.

Example 2.1. *Maximize $4x - 8x^2$. Consider initial point as zero and step length as 0.1.*

Solution:

From x = 0.0, either a movement is to be made in positive direction or negative direction.

At $\quad x = 0,\ f(x) = 4x - 8x^2 = 0$

As the step length is 0.1, value of x in negative direction is 0 – 0.1 = –0.1 and its value in positive direction is 0 + 0.1 = + 0.1

At $\quad x = -0.1,\ f(x) = -0.48$

At $\quad x = +0.1,\ f(x) = +0.32$

As the objective is to maximize and 0.32 > – 0.48, it is reasonable to proceed in positive direction.

x	$f(x)$
0	0
0.1	0.32
0.2	0.48
0.3	0.48

$f(0.3)$ is not greater than $f(0.2)$, therefore an exact optimum may lie in the close range [0.2, 0.3]. This is searched using smaller step length say 0.01 from $x = 0.2$ after finding suitable direction.

x	$f(x)$
0.21	0.4872
0.22	0.4928
0.23	0.4968
0.24	0.4992
0.25	0.5
0.26	0.4992

$f(0.26)$ is not greater than $f(0.25)$, therefore optimum function value is 0.5 corresponding to optimum $x^* = 0.25$.

Example 2.2. *Minimize $8x^2 - 5.44x$. Use initial value of $x = 0.5$ and step length = 0.1.*

Solution. As the objective is to minimize, lower function value is preferred.

$$f(0.5) = -0.72$$

In order to find out suitable direction, value of x is increased and decreased by step length = 0.1.

$$f(0.6) = -0.384$$

$$f(0.4) = -0.896$$

as the function has a tendency to decrease in negative direction from the initial value, the search is made as follows :

x	$f(x)$
0.5	– 0.72
0.4	– 0.896
0.3	– 0.912
0.2	– 0.768

As $f(0.2)$ is not less than $f(0.3)$, the search is stopped at this stage. It is restarted from $x = 0.3$ with smaller step length, say 0.01. In order to find an appropriate direction for movement,

At $x = 0.3 - 0.01 = 0.29, f(x) = f(0.29) = -0.9048$

At $x = 0.3 + 0.01 = 0.31, f(x) = f(0.31) = -0.9176$

Further improvement (from the minimization point of view) is in positive direction, therefore the process in the close range of [0.3, 0.4] is repeated as follows :

x	$f(x)$
0.31	– 0.9176
0.32	– 0.9216
0.33	– 0.9240
0.34	– 0.9248
0.35	– 0.9240

As $f(0.35)$ is not less than $f(0.34)$, the process is stopped at this stage. From $x = 0.34$, consider step length as 0.001.

$$f(0.339) = -0.92479$$

$$f(0.341) = -0.92479$$

As compared to $f(0.34) = -0.9248$, no further improvement is possible in either direction, therefore $f(0.34) = -0.9248$ is optimum function value corresponding to $x^* = 0.34$.

In many practical situations, an idea related to the range in which an optimum variable may lie is readily available and the search is restricted to that range only. For example, the pitch circle diameter of a gear to be assembled in any equipment may not be greater than 200 mm. Similarly this diameter may not be less than 50 mm depending on several factors such as desired output range, convenience in manufacturing, accuracy and capacity of available machine for fabricating the gear, etc. The search process may be restricted to the range [50, 200] mm from the beginning.

Example 2.3. *A manufacturing industry is in the business of producing multiple items in a cycle and each item is produced in every cycle. Its total relevant annual cost is the sum of machine setup cost and inventory holding cost. Total cost function is estimated to be*

$$f(x) = 1150x + 250/x,$$

where x is the common cycle time in year. Objective is to minimize the total cost function f(x) and to evaluate optimum x for implementation. Apply the search procedure as discussed before considering x = 0.45 year and step length as 0.01 in the first stage of the search.

Also solve the problem by testing whether the function is convex and using the convexity property. Comment on the importance of the search procedures.

Solution.

$$f(0.45) = 1073.06$$

Since $f(0.46) = 1072.47 < f(0.44) = 1074.18$, and as compared to $f(0.45) = 1073.06$, cost improvement is possible in positive direction.

x	$f(x)$
0.46	1072.47
0.47	1072.41
0.48	1072.83

Starting from 0.47 again and using smaller step length = 0.001, negative direction is chosen, and

x:	0.469	0.468	0.467	0.466	0.465
$f(x)$:	1072.40	1072.39	1072.3819	1072.3807	1072.384

Therefore the optimal cycle time x is obtained as 0.466 year and total relevant cost as Rs. 1072.38.

In order to test whether the function is convex,

$$\frac{d^2 f(x)}{dx^2} = \frac{500}{x^3} > 0$$ for positive x and therefore it is a convex function.

Using the convexity property, optimum may be obtained by differentiating $f(x)$ with respect to x and equating to zero,

$$\frac{d\,f(x)}{dx} = 1150 - \frac{250}{x^2} = 0$$

or

$$x^* = \sqrt{\frac{250}{1150}} = 0.466 \text{ year}$$

and optimum value of total cost, $f(x^*) =$ Rs. 1072.38. The results are similar to those obtained using the search procedure.

However in many cases, it is difficult to test whether the function is convex or concave. In other words, it is difficult to differentiate any function. Properties of convex or concave function may not be used to evaluate the optimum if it is not possible to ascertain the type of function. In such situations, the search procedures become very important in order to compute the optimal value.

2.2 METHOD OF GOLDEN SECTION

This method is suitable for the situation in which a maximum lies in the given range and the function is unimodal in that range. Consider a function $f(x)$ as shown in Fig. 2.2. A known range of $[x_1, x_2]$ is available in which an optimum x^* lies.

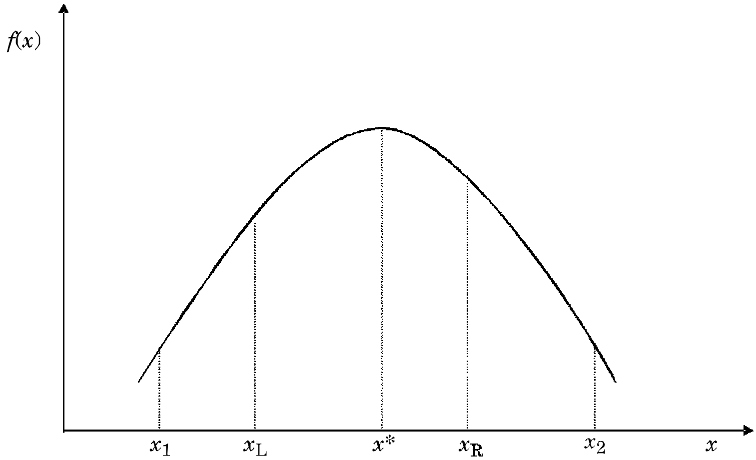


Fig. 2.2: Optimum x^* in the range $[x_1, x_2]$.

Let there be two points x_L and x_R in the range $[x_1, x_2]$ such that,

x_L = Point on the left hand side

x_R = Point on the right hand side

Three possibilities may arise

$f(x_L) < f(x_R)$, $f(x_L) = f(x_R)$, and $f(x_L) > f(x_R)$, which are categorized into two:

(*i*) $f(x_L) < f(x_R)$

(*ii*) $f(x_L) \geq f(x_R)$

Take the first case i.e. when $f(x_L) < f(x_R)$.

This may be true in two situations:

(*a*) When x_L and x_R are on either side of the optimum x^*, as shown in Fig. 2.2.

(*b*) when x_L and x_R are on one side of the optimum as shown in Fig. 2.3.

Following statement is applicable in both the situations :

"Optimum should lie in the range $[x_L, x_2]$ if $f(x_L) < f(x_R)$".

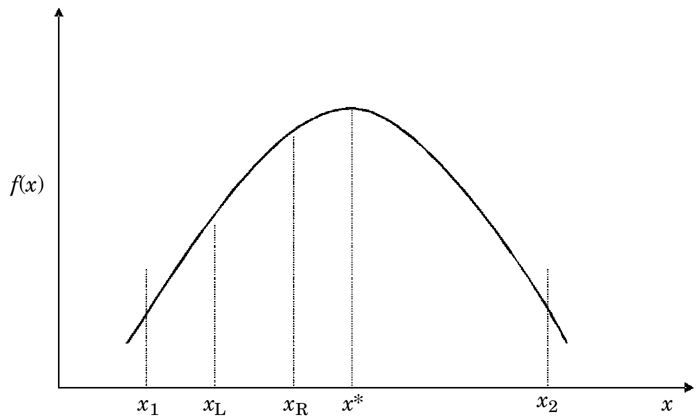


Fig. 2.3: x_L and x_R on one side of the optimum $[f(x_L) < f(x_R)]$.

Now take the second case i.e. when $f(x_L) \geq f(x_R)$.

This may be true in the three situations as shown in Fig. 2.4.

Fig. 2.4(*a*) – x_L and x_R on either side of the optimum and $f(x_L) = f(x_R)$.

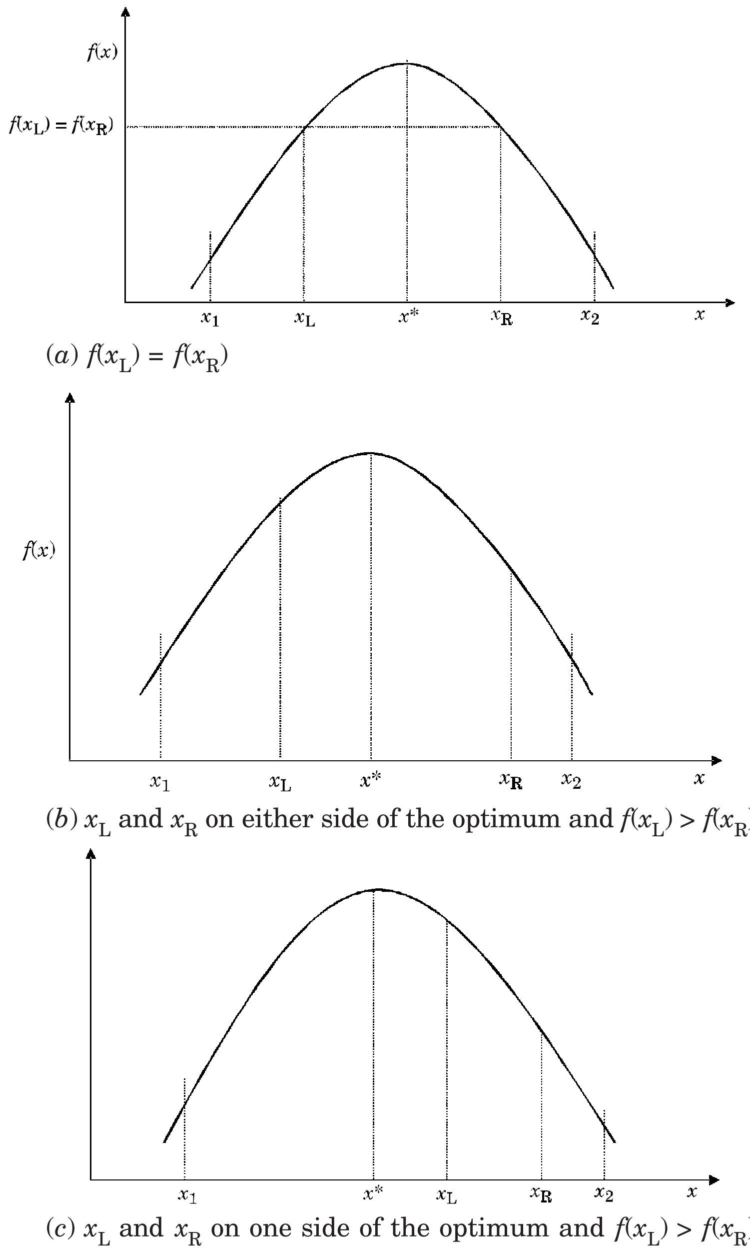
Fig. 2.4(*b*) – x_L and x_R on either side of the optimum and $f(x_L) > f(x_R)$.

Fig. 2.4(*c*) – x_L and x_R on one side of the optimum and $f(x_L) > f(x_R)$.

Following statement is applicable in all the three situations:

"Optimum should lie in the range $[x_1, x_R]$ if $f(x_L) = f(x_R)$".

Observe this and the previous statement. Out of x_L and x_R, only one is changing for the range in which an optimum may lie. This discussion is useful for implementing an algorithm for method of golden section.



(*a*) $f(x_L) = f(x_R)$

(*b*) x_L and x_R on either side of the optimum and $f(x_L) > f(x_R)$

(*c*) x_L and x_R on one side of the optimum and $f(x_L) > f(x_R)$

Fig. 2.4: Various situations for $f(x_L) \geq f(x_R)$

2.2.1 Algorithm

The algorithm for method of golden section is illustrated by Fig. 2.5 where $[x_1, x_2]$ is the range in which maximum x^* lies, and

$$r = 0.5\ (\sqrt{5} - 1) = 0.618034 \simeq 0.618$$

This number has certain properties which may be observed later.

After initialization as mentioned in Fig. 2.5, following steps are followed:

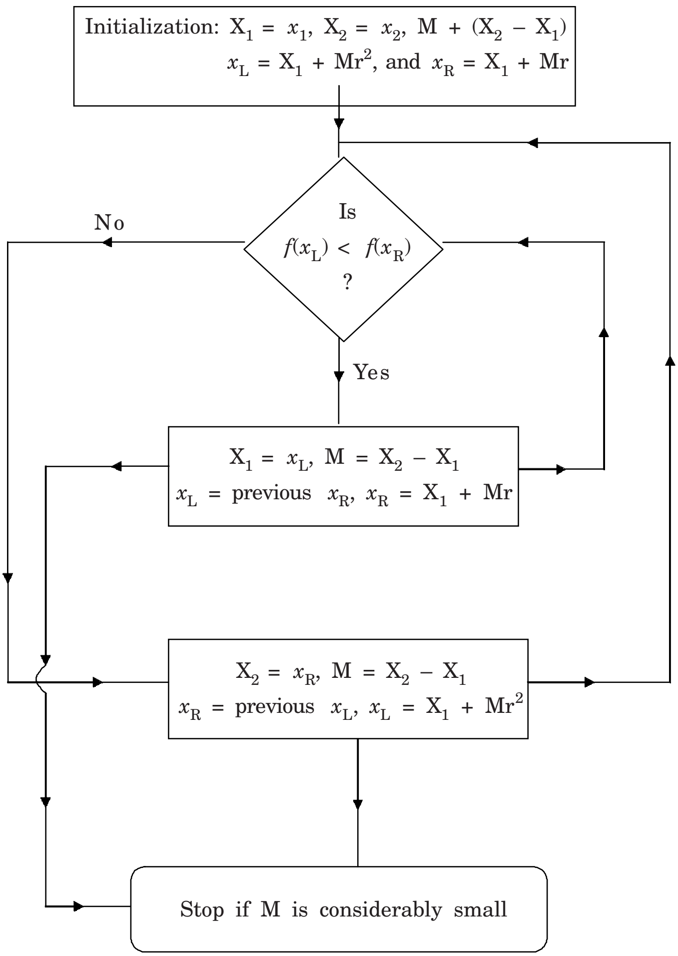


Fig. 2.5: Algorithm for the method of golden section.

Step 1 – $f(x_L)$ is compared with $f(x_R)$ and depending on this comparison, go to either step 2 or step 3.

Step 2 – if $f(x_L) < f(x_R)$, $X_1 = x_L$ and $M = X_2 - X_1$

x_L = previous x_R and $x_R = X_1 + Mr$

Go to step 1.

Step 3 – If $f(x_L) \geq f(x_R)$, $X_2 = x_R$ and $M = X_2 - X_1$

x_R = previous x_L and $x_L = X_1 + Mr^2$

Go to step 1.

The procedure is stopped if value of M is considerably small.

Example 2.4. *Consider the problem of example 2.1 in which $f(x) = 4x - 8x^2$, is maximized. Implement the method of golden section. Use the initial range as [0, 0.5] in which optimum lies.*

Solution. Now $[x_1, x_2] = [0, 0.5]$

Initialization: $X_1 = x_1 = 0$

$X_2 = x_2 = 0.5$

$M = X_2 - X_1 = 0.5 - 0 = 0.5$

$x_L = X_1 + Mr^2 = 0 + (0.5 \times 0.6182) = 0.19$

$x_R = X_1 + Mr = 0 + (0.5 \times 0.618) = 0.31$

Iteration 1:

Step 1 – $f(x_L) = f(0.19) = 0.47$

$f(x_R) = f(0.31) = 0.47$

As the condition $f(x_L) \geq f(x_R)$ is satisfied step 3 is applicable.

Step 3 – $X_2 = 0.31$ and $M = 0.31 - 0 = 0.31$

$x_R = 0.19$ and $x_L = 0 + (0.31 \times 0.6182) = 0.12$

Iteration 2:

Step 1 – $f(x_L) = f(0.12) = 0.36$

$f(x_R) = f(0.19) = 0.47$

As $f(x_L) < f(x_R)$, step 2 is applicable.

Step 2 – $X_1 = 0.12$ and $M = X_2 - X_1$

$= 0.31 - 0.12 = 0.19$

$x_L = 0.19$ and x_R

$= 0.12 + (0.19 \times 0.618) = 0.24$

It may be observed that only one value out of x_L and x_R is really changing in each iteration. Either x_L or x_R will take the previous value of x_R or x_L respectively.

At any iteration i, value of M = r. (value of M at iteration, i–1)

For example, M = 0.19 in iteration 2, which is equal to 0.618 (value of M in iteration 1) or 0.618 × 0.31 = 0.19.

Alternatively, value of M (or the range in which optimum lies) at any iteration $i = (x_2 - x_1)\, r^i$

where i = 1, 2, 3, ...

At iteration 1, M = (0.5 – 0) × 0.618 = 0.31

At iteration 2, M = (0.5 – 0) × 0.618^2 = 0.19

Iterative process is continued until M is considerably small.

Iteration 3:

Step 1 : $f(x_L) = f(0.19) = 0.47$

$f(x_R) = f(0.24) = 0.50$

Step 2 : $X_1 = 0.19$ and M = 0.31 – 0.19 = 0.12

$x_L = 0.24$ and x_R

$= 0.19 + (0.12 \times 0.618) = 0.26$

Iteration 4:

Step 1 : $f(x_L) = f(0.24) = 0.50$

$f(x_R) = f(0.26) = 0.50$

Step 3 : $X_2 = 0.26$, M = 0.26 – 0.19 = 0.07

$x_R = 0.24$, $x_L = 0.19 + (0.07 \times 0.618^2) = 0.22$

Iteration 5:

Step 1 : $f(x_L) = f(0.22) = 0.49$

$f(x_R) = f(0.24) = 0.50$

Step 2 : $X_1 = 0.22$, M = 0.26 – 0.22 = 0.04

$x_L = 0.24$, $x_R = 0.22 + (0.04 \times 0.618) = 0.24$

The process may be continued to any desired accuracy. At present, value of M = 0.04 with $[X_1, X_2] = [0.22, 0.26]$, which indicates that optimum lies between 0.22 and 0.26. Considering it a narrow range, even an average works out to be 0.24 which is very close to exact optimum $x^* = 0.25$.

At any iteration i, $x_L - X_1 = X_2 - x_R$.

For example, at iteration 3,

$$0.24 - 0.19 = 0.31 - 0.26 = 0.05$$

2.3 QUADRATIC INTERPOLATION

If it is possible to approximate any function by a quadratic function or it is difficult to differentiate it, then the quadratic function is analyzed in order to obtain a minimum. The minimum thus obtained is substituted in the original function which is to be minimized and the process is continued to attain the desired accuracy.

A quadratic function, $f(x) = ax^2 + bx + c$ ...(2.1)

Three points x_1, x_2 and x_3 are selected, and

$$f(x_1) = ax_1^2 + bx_1 + c \quad ...(2.2)$$

$$f(x_2) = ax_2^2 + bx_2 + c \quad ...(2.3)$$

$$f(x_3) = ax_3^2 + bx_3 + c \quad ...(2.4)$$

solving these three equations (2.2), (2.3) and (2.4), values of a and b are obtained as follows :

$$a = -\left(\frac{(x_1 - x_2) f(x_3) + (x_2 - x_3) f(x_1) (x_3 - x_1) f(x_2)}{(x_1 - x_2)(x_2 - x_3)(x_3 - x_1)}\right)$$

...(2.5)

$$b = \left(\frac{(x_1^2 - x_2^2) f(x_3) + (x_2^2 - x_3^2) f(x_1) (x_3^2 - x_1^2) f(x_2)}{(x_1 - x_2)(x_2 - x_3)(x_3 - x_1)}\right)$$

...(2.6)

In order to obtain minimum of equation (2.1),

$$\frac{d\,f(x)}{dx} = 0$$

or $$2ax + b = 0$$

or Minimum $$x^* = -b/2a$$

Substituting the values of a and b from equations (2.5) and (2.6) respectively,

$$x^* = \frac{1}{2}\left(\frac{(x_1^2 - x_2^2)\, f(x_3) + (x_2^2 - x_3^2) f(x_1)\, (x_3^2 - x_1^2) f(x_2)}{(x_1 - x_2)\, f(x_3)\, (x_2 - x_3)\, f(x_1)\, (x_3 - x_1)\, f(x_2)}\right) \quad ...(2.7)$$

This minimum x^* is used in the iterative process. Three points x_1, x_2 and x_3 as well as their function values are needed to determine x^*.

An initial approximate point x_1 is given,

and $x_2 = x_1 + \Delta$

where Δ = step length

As the objective is to minimize, a third point x_3 is selected as follows :

(*i*) $x_3 = x_1 - \Delta$, if $f(x_1) < f(x_2)$

(*ii*) $x_3 = x_2 + \Delta = x_1 + 2\Delta$, if $f(x_2) < f(x_1)$

Example 2.5. *Obtain the minimum of the following function using quadratic interpolation*

$$f(x) = 1200x + 300/x$$

Initial approximate point may be assumed as $x_1 = 0.3$ *and step length* $\Delta = 0.1$.

Solution.

Iteration 1:

Now $x_1 = 0.3$ and $f(x_1) = 1360$

$x_2 = x_1 + \Delta = 0.3 + 0.1 = 0.4$ and $f(x_2) = 1230$

As $f(x_2) < f(x_1)$, $x_3 = x_1 + 2\Delta = 0.5$

And $f(x_3) = 1200$

From equation (2.7), $x^* = 0.48$

And $f(x^*) = 1201$

Iteration 2:

$x^* = 0.48$ may replace the initial value x_1

Now $x_1 = 0.48$, $f(x_1) = 1201$

$x_2 = 0.48 + 0.1 = 0.58$, $f(x_2) = 1213.24$

As $f(x_1) < f(x_2)$, $x_3 = x_1 - \Delta = 0.48 - 0.1 = 0.38$

and $f(x_3) = 1245.47$

From equation (2.7), $x^* = 0.508$

And $f(x^*) = 1200.15$

In order to achieve the desired accuracy, the process may be continued until the difference between consecutive values of x^* becomes very small.

2.4 NEWTON RAPHSON METHOD

Consider $x = a$ as an initial approximate value of the optimum of a function $f(x)$ and $(a + h)$ as an improved value. In order to find out the value of h, expand $f(a + h)$ using Taylor's theorem and ignore higher powers of h. Further following notations may be assumed for convenience

$f^1(x)$ = first order derivative

$f^{11}(x)$ = second order derivative

Now $f(a + h) = f(a) + hf^1(a)$

For the minima, $f^1(a + h) = f^1(a) + hf^{11}(a) = 0$

or
$$h = \frac{-f^1(a)}{f^{11}(a)}$$

$$\text{Next value of } x = a - \frac{f^1(a)}{f^{11}(a)} \tag{2.8}$$

Example 2.6. *Use the Newton Raphson method in order to solve the problem of Example 2.5 in which,*

$$f(x) = 1200x + 300/x$$

obtain the minimum considering initial value of $x = 0.3$.

Solution.

Iteration 1:

$$f^1(x) = 1200 - 300/x^2 \tag{2.9}$$

$$f^{11}(x) = 600/x^3 \tag{2.10}$$

Now $a = 0.1$

From (2.9), $f^1(a) = -2133.33$

From (2.10), $f^{11}(a) = 22222.22$

From (2.8), new value of $x = 0.3 + \frac{2133.33}{22222.22}$

$= 0.396$

This value is used as a in the next iteration.

Iteration 2:

Now $a = 0.396$

Again using (2.9) and (2.10),

$f^{1}(a) = -\ 713.07$

$f^{11}(a) = 9661.97$

From (2.8), new value of $x = 0.396 + \frac{713.07}{9661.97}$

$= 0.47$

Iteration 3:

Now $a = 0.47$

$f^{1}(a) = -\ 158.08,$

$f^{11}(a) = 5779.07$

New value of $x = 0.47 + 0.027 = 0.497$

The process is continued to achieve any desired accuracy.

In a complex manufacturing-inventory model, all the shortage quantities are not assumed to be completely backordered. A fraction of shortage quantity is not backlogged. Total annual cost function is formulated as the sum of procurement cost, setup cost, inventory holding cost and backordering cost. After substituting the maximum shortage quantity (in terms of manufacturing lot size) in the first order derivative of this function with respect to lot size and equated to zero, a quartic equation is developed, which is in terms of single variable, *i.e.*, lot size. Newton-Raphson method may be successfully applied in order to obtain the optimum lot size using an initial value as the batch size for the situation in which all the shortages are assumed to be completely backordered.

If any equation is in the form, $f(x) = 0$, and the objective is to find out x which will satisfy this, then the Newton-Raphson method may be used as follows:

$$f(a + h) = f(a) + hf^1(a) = 0$$

or
$$h = \frac{-f(a)}{f^1(a)} \tag{2.11}$$

and the next value of $x = a - \dfrac{f(a)}{f^1(a)}$ (2.12)

This value of x is used in the next iteration as an approximate value, a. The process is continued until the difference between two successive values of x becomes very small.

3

UNCONSTRAINED AND CONSTRAINED OPTIMIZATION

One variable optimization problems were discussed in the previous chapter. There are many situations in which more than one variable occur in the objective function. In an unconstrained multivariable problem, the purpose is to optimize $f(X)$ and obtain,

$$X = \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{pmatrix}$$

without imposing any constraint.

In a constrained multivariable problem, the objective is to optimize $f(X)$, and get,

$$X = \begin{pmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{pmatrix}$$

subject to the inequality constraints such as,

$$p_i(X) \leq 0, \; i = 1, 2, ..., k$$
$$q_i(X) \geq 0, \; i = 1, 2, ..., l$$

and/or, equality constraints,

$$r_i(X) = 0, \; i = 1, 2, \ldots \; m$$

3.1 DICHOTOMOUS SEARCH

In this search procedure, the aim is to maximize a function $f(x)$ over the range $[x_1, x_2]$ as shown in Fig. 3.1

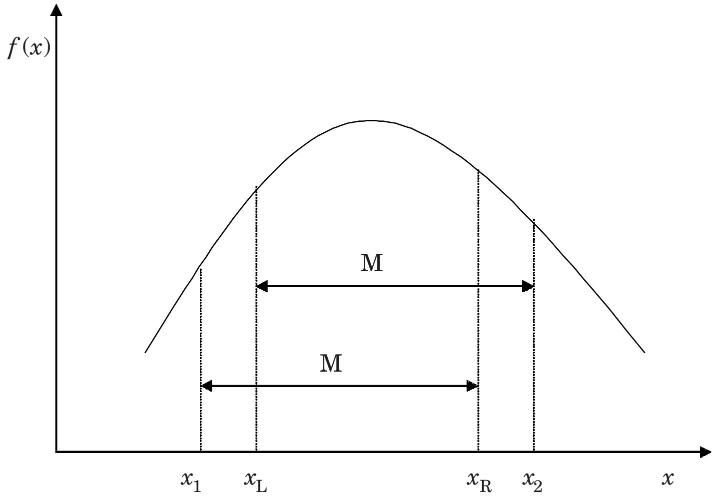


Fig. 3.1: Dichotomized range $[x_1, x_2]$.

Two points x_L and x_R are chosen in such a way, that,

$$M = x_R - x_1 = x_2 - x_L \quad \ldots(3.1)$$

Consider $\Delta = x_R - x_L$ ...(3.2)

Solving equations (3.1) and (3.2),

$$x_L = \frac{x_1 + x_2 - \Delta}{2} \quad \ldots(3.3)$$

$$x_R = \frac{x_1 + x_2 + \Delta}{2} \quad \ldots(3.4)$$

Value of Δ is suitably selected for any problem and x_L as well as x_R are obtained. After getting the function values $f(x_L)$and $f(x_R)$,

(*i*) If $f(x_L) < f(x_R)$, then the maximum will lie in the range $[x_L, x_2]$.

(*ii*) If $f(x_L) > f(x_R)$, then the maximum will lie in the range $[x_1, x_R]$.

(*iii*) If $f(x_L) = f(x_R)$, then the maximum will lie in the range $[x_L, x_R]$.

Using appropriate situation, the range for the search of a maximum becomes narrow in each iteration in comparison with previous iteration. The process is continued until the range, in which optimum lies, *i.e.* $(x_2 - x_1)$, is considerably small.

Example 3.1. *Maximum $f(x) = 24x - 4x^2$ using dichotomous search. Initial range $[x_1, x_2] = [2, 5]$. Consider $\Delta = 0.01$ and stop the process if considerably small range (in which maximum lies) is achieved, i.e. $(x_2 - x_1) = 0.1$.*

Solution.

Iteration 1:

From equations (3.3) and (3.4),

$$x_L = \frac{2 + 5 - 0.01}{2} = 3.495$$

$$x_R = \frac{2 + 5 + 0.01}{2} = 3.505$$

$$f(x_L) = 35.0199$$

$$f(x_R) = 34.9799$$

As $f(x_L) > f(x_R)$, then the maximum lies in the range $[x_1, x_R] = [2, 3.505]$.

For the next iteration $[x_1, x_2] = [2, 3.505]$.

Iteration 2:

Again using equations (3.3) and (3.4),

$$x_L = \frac{2 + 3.505 - 0.01}{2} = 2.7475$$

$$x_R = \frac{2 + 3.505 + 0.01}{2} = 2.7575$$

and

$$f(x_L) = 35.7449$$

$$f(x_R) = 35.7648$$

As $f(x_L) < f(x_R)$, then the maximum lies in the range $[x_L, x_2] = [2.7475, 3.505]$.

Iteration 3:

Now $[x_1, x_2] = [2.7475, 3.505]$.

$$x_L = 3.12125, f(x_L) = 35.9412$$
$$x_R = 3.13125, f(x_R) = 35.9311$$

$f(x_L) > f(x_R)$, range for the maximum = $[x_1, x_R]$ = [2.7475, 3.13125].

Iteration 4:

$$[x_1, x_2] = [2.7475, 3.13125]$$
$$x_L = 2.934375, f(x_L) = 35.9828$$
$$x_R = 2.944375, f(x_R) = 35.9876$$

$f(x_L) < f(x_R)$, and for the next iteration,

$$[x_1, x_2] = [x_L, x_2] = [2.934375, 3.13125]$$

Iteration 5:

$$x_L = 3.0278, f(x_L) = 35.9969$$
$$x_R = 3.0378, f(x_R) = 35.9943$$

$f(x_L) > f(x_R)$, and maximum will lie in the range $[x_1, x_R]$ = [2.934375, 3.0378].

Iteration 6:

$$[x_1, x_2] = [2.934375, 3.0378]$$
$$x_L = 2.9811, f(x_L) = 35.9986$$
$$x_R = 2.9911, f(x_R) = 35.9997$$

$f(x_L) < f(x_R)$, and a range for the maximum = $[x_L, x_2]$ = [2.9811, 3.0378].

The process will be stopped as $[x_1, x_2]$ = [2.9811, 3.0378] and $(x_2 - x_1) = 0.0567$, which is < 0.1.

Consider the average of this small range as maximum, *i.e.* (2.9811 + 3.0378)/2 = 3.009. This is very near to the exact optimum *i.e.* 3.

3.2 UNIVARIATE METHOD

This is suitable for a multivariable problem. One variable is chosen at a time and from an initial point, it is increased or decreased depending on the appropriate direction in order to minimize the function. An optimum step length may be determined as illustrated in the example. Similarly each variable is changed, one at a time, and that is why the name "univariate method". One cycle is said to complete when all the variables have been changed. The next cycle will start from this position.

The procedure is stopped when further minimization is not possible with respect to any variable.

Consider a function $f(x_1, x_2)$. Optimum x_1 and x_2 are to be obtained in order to minimize $f(x_1, x_2)$. Fig. 3.2 is a representation of the univariate method.

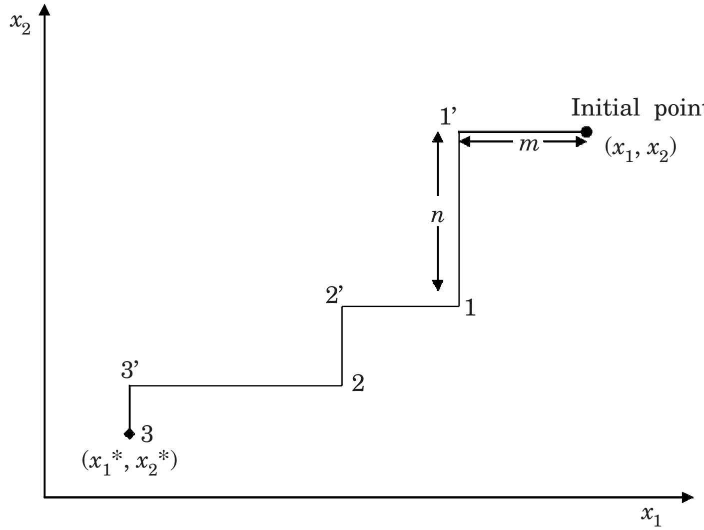


Fig. 3.2 Changing one variable at a time.

From the initial point (x_1, x_2), only x_1 is varied. In order to find out suitable direction, probe length is used. Probe length is a small step length by which the current value of any variable is increased or decreased, so that appropriate direction may be identified. After determining optimum step length m (assuming negative direction), value of x_1 is changed to that corresponding to point 1'. From point 1' repeat the process with respect to variable x_2, without changing x_1. Cycle 1 is completed at point 1 and from this position of (x_1, x_2) cycle 2 will start, which is completed at point 2. The optimum (x_1^*, x_2^*) correspond to point 3 and the objective is to reach there.

The process is continued till there is no chance of further improvement with respect to any of the variables.

Example 3.2. *Minimize the following function for the positive values of* x_1 *and* x_2*, using univariate method*

$$f(x_1, x_2) = 9x_1 + \frac{3}{x_1} + \frac{3x_1^2}{x_2} + 27x_2$$

Probe length may be considered as 0.05 and an initial point as (1, 1).

Solution. As the initial point is (1, 1), $f(1, 1) = 42$

Cycle 1.

(*a*) **Vary x_1 :**

As the probe length is 0.05,

$$f(0.95, 1) = 41.42$$

and $$f(1.05, 1) = 42.61$$

Since the objective is to minimize, and 41.42 < 42, negative direction is chosen. Considering m as step length,

$$f[(1 - m), 1] = 9(1 - m) + \frac{3}{(1 - m)} + 3(1 - m)^2 + 27$$

m can be optimized by differentiating this equation and equating to zero. In case, it is difficult to differentiate or evaluate m, simple search procedure may be adopted to obtain approximate optimal step length m.

Now $$\frac{df}{dm} = 0 \text{ shows}$$

$$2m^3 - 9m^2 + 12m - 4 = 0$$

$m = 0.5$ can be obtained using simple search technique or other suitable method.

New value of $x_1 = 1 - m = 1 - 0.5 = 0.5$

And $f(x_1, x_2) = f(0.5, 1) = 38.25$

(*b*) **Vary x_2:**

Using the probe length 0.05,

$$f(0.5, 0.95) = 36.94$$

and $$f(0.5, 1.05) = 39.56$$

Negative direction is selected for varying x_2, and considering step length n,

$$f(0.5, (1 - n)] = 37.5 + \frac{0.75}{(1 - n)} - 27n$$

$$\frac{df}{dn} = 0 \text{ shows,}$$

$$n = 0.83$$

New value of $x_2 = 1 - n = 1 - 0.83 = 0.17$

and $f(x_1, x_2) = x(0.5, 0.17) = 19.50$

This is corresponding to point 1 as shown in Fig. 3.2.

Cycle 2

(*a*) **Vary x_1:**

$f(0.55, 0.17) = 20.33$

$f(0.45, 0.17) = 18.88 < f(0.5, 0.17) = 19.50$

Again negative direction is chosen and

$$f[(0.5 - m), 0.17] = 9(0.5 - m) + \frac{3}{(0.5 - m)} + \frac{3(0.5 - m^2)}{0.17} + 4.59$$

$$\frac{df}{dm} = 0 \text{ shows}$$

$$11.76m + \frac{1}{(0.5 - m)^2} - 8.88 = 0 \qquad \text{...(3.5)}$$

Value of m needs to be obtained which will satisfy above equation. As we are interested in positive values of variables, $m < 0.5$. Considering suitable fixed step size as 0.1, values of the L.H.S. of equation obtained in terms of m are as follows:

m	**L.H.S. value for equation (3.5)**
0.4	95.824
0.3	19.648
0.2	4.58
0.1	–1.45

Value of m should be between 0.1 and 0.2. Using smaller step size as 0.01,

m	**L.H.S. value for equation (3.5)**
0.11	–1.01
0.12	–0.54
0.13	–0.05
0.14	0.48

Therefore, approximate value for m is considered as 0.13 and new value of $x_1 = 0.5 - 0.13 = 0.37$.

$f(0.37, 0.17) = 18.44$, which is less than the previous value i.e. 19.50 corresponding to $f(0.5, 0.17)$.

(*b*) **Vary x_2:**

$$f(0.37, 0.22) = 19.24$$

$$f(0.37, 0.12) = 18.10$$

Negative direction is chosen, and

$$f[0.37, (0.17 - n)] = 11.44 + \frac{0.41}{(0.17 - n)} + 27\,(0.17 - n)$$

$$\frac{df}{dn} = 0 \text{ shows}$$

$$n = 0.05$$

and new value of $x_2 = 0.17 - 0.05 = 0.12$

$$f(x_1, x_2) = f(0.37, 0.12) = 18.10$$

The procedure is continued for any number of cycles till improvement in function values are observed by varying x_1 and x_2 each. The exact optimum is $[x_1^*, x_2^*] = [1/3, 1/9]$.

3.3 PENALTY FUNCTION METHOD

This method is suitable for the constrained optimization problem such as,

Minimize $f(X)$

Subject to inequality constraints in the form,

$$p_i(X) \leq 0, \; i = 1, 2, ..., k$$

and obtain

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ x_n \end{pmatrix}$$

This is transformed to an unconstrained minimization problem by using penalty parameter and constraints in a suitable form. The new function is minimized by using a sequence of the values of penalty parameter. The penalty function method is also called as sequential unconstrained minimization technique (SUMT).

3.3.1 Interior Penality Function Method

A new function Φ is introduced as follows:

$$\Phi(X, r) = f(X) - r \sum_{i=1}^{k} \frac{1}{p_i(X)} \quad ...(3.6)$$

whereas r is the non-negative constant known as the penalty parameter. Starting from the suitable initial value of r, decreasing sequence of the values of penalty parameter, is used to minimize $\Phi(X, r)$ and eventually $f(X)$, in the interior penalty function method.

Example 3.3.

$$\textit{Minimize } f(X) = f(x_1, x_2) = x_1 + \frac{(4 + x_2)^3}{3}$$

$$\textit{subject to } x_1 \geq 1$$

$$x_2 \geq 4$$

Use the initial value of penalty parameter r as 1 and multiplication factor as 0.1 in each successive iteration.

Solution. As the constraints need to be written in "$p_i(X) \leq 0$" form, these are

$$1 - x_1 \leq 0$$

$$4 - x_2 \leq 0$$

In order to obtain a new function Φ, equation (3.6) is used. Sum of the reciprocal of L.H.S. of the constraints, is multiplied with penalty parameter r, and then this is deducted from function $f(X)$.

Now $\Phi(X, r) = \Phi(x_1, x_2, r)$

$$= x_1 + \frac{(4 + x_2)^3}{3} - r\left(\frac{1}{(1 - x_1)} + \frac{1}{(4 - x_2)}\right) \quad ...(3.7)$$

In order to minimize Φ, this is differentiated partially with respect to x_1 and x_2, and equated to zero.

$$\frac{\partial \Phi}{\partial x_1} = 1 - \frac{r}{(1 - x_1)^2} = 0$$

or $$x_1^* = 1 - \sqrt{r} \quad ...(3.8)$$

$$\frac{\partial \Phi}{\partial x_2} = (4 + x_2)^2 - \frac{r}{(4 - x_2)^2} = 0$$

or $$x_2^* = [16 - \sqrt{r}\,]^{1/2} \qquad ...(3.9)$$

x_1 and x_2 are obtained for any value of r from equations (3.8) and (3.9) respectively. These are substituted in equation (3.7) in order to get Φ^* and similarly f(X) or $f(x_1, x_2)$ are obtained for decreasing sequence of r. As the multiplication factor is 0.1, this is multiplied with the previous value of r in each iteration. Calculations are shown in Table 3.1.

Table 3.1: Calculations

Iteration, i	*r*	x_1^*	x_2^*	Φ^*	$f(x_1, x_2)$
1	1	0	3.87	153.79	162.48
2	0.1	0.68	3.96	165.99	168.79
3	0.01	0.9	3.99	169.83	170.93
4	0.001	0.97	3.996	171.10	171.38
5	0.0001	0.99	3.999	171.48	171.59

Values of Φ as well as f are increasing. This is because $\sqrt{r}$ has been considered as positive. Taking it as negative, a decreasing trend for Φ and f would have been obtained, along with $\Phi > f$.

As shown in Table 3.1, with increased number of iterations, values of Φ and f are almost similar. The process may be continued to any desired accuracy and it is approaching to exact optimum $f^*(x_1, x_2) = 171.67$ along with $x_1^* = 1$ and $x_2^* = 4$.

3.4 INDUSTRIAL APPLICATION

The objectives to be achieved in different area of any industry/business are of many kinds. Some of these are as follows:

(*a*) Manufacturing planning and control

(*b*) Quality management

(*c*) Maintenance planning

(*d*) Engineering design

(*e*) Inventory management

Machining of the components are often needed in the manufacturing of a product. The aim may be

(*i*) the least machining cost per component,

(*ii*) the minimum machining time per component, or a suitable combination of both

The decision variables may include cutting speed, feed and depth of cut, optimum values of which are to be evaluated subject to the operational constraints. The operational constraints are of various types such as tool life, surface finish, tolerance etc.

Different products are designed in any industry depending on their quality requirements and final application. If this product is a machine component, then its suitability from maintenance point of view becomes significant. For many of the items, maximizing their value is also necessary.

Gears are used for transmission of the motion. Depending on the parameters such as speed and power to be transmitted, suitable material for the gear is selected. The objective function may include

(*i*) minimize the transmission error,

(*ii*) maximize the efficiency, and

(*iii*) minimize the weight.

Various constraints are imposed on the O.F. In addition to the space constraint, the induced stresses such as crushing, bending and shear stress must be less than or equal to the respective allowable crushing, bending and shear stress of the selected material.

In a business application, while marketing the finished products, price per unit is not constant. Price per unit may be lower if larger quantities are to be sold. Total sales revenue is evaluated by multiplying quantity with price function. Profit is computed by subtracting total cost from total revenue. This profit is a nonlinear function in terms of quantity, which needs to be maximized subject to operational constraints.

The inventories mainly consist of raw material, work-in-process and finished products. The scope of the inventory management lies, in most of the cases, in the minimization of total procurement cost, total manufacturing cost, and the costs incurred in the supply chain. Objective function is formulated as the total relevant cost in many of the situations. The production-inventory model is optimized subject to certain constraints. Depending on the problem, there may be constraints on-

(*i*) storage space

(*ii*) capacity

(*iii*) capital

(*iv*) machine setup and actual production time.

(*v*) shelf life.

Example 3.4. *A manufacturing industry is engaged in the batch production of an item. At present, its production batch size,* $x_1 = 4086$ *units without allowing any shortage quantity in the manufacturing cycle. Now the industry is planning to allow some shortages in the manufacturing system. However it will not allow the maximum shortage quantity* x_2*, to be more than 4 units in the production-inventory cycle. Total relevant cost function,* $f(x_1, x_2)$ *is estimated to be as follows:*

$$f(x_1, x_2) = 1370.45 \frac{x_2^2}{x_1} + 0.147x_1 - 2x_2 + \frac{2448979.6}{x_1}$$

where x_1 = *Production batch size*

and x_2 = *Maximum shortage quantity.*

*Discuss the proced*ure briefly in order to minimize $f(x_1, x_2)$ subject to $x_2 \leq 4$.

Solution. As discussed in section 3.3, the constraint is written as,

$$x_2 - 4 \leq 0$$

and function,

$$\Phi = 1370.45 \frac{x_2^2}{x_1} + 0.147x_1 - 2x_2 + \frac{2448979.6}{x_1} - r\left(\frac{1}{x^2 - 4}\right) \qquad ...(3.10)$$

A combination of interior penalty function method and univariate method may be used. For different values of penalty parameter r, univariate method may be applied in order to obtain the appropriate x_1 and x_2 at that stage. Following initial values are considered :

r = 1 and multiplication factor = 0.01

x_1 = 4086 units

x_2 = 1 unit

and after substituting,

Φ = 1198.6694

f = 1198.34

r = 1:

Cycle 1:

One variable is changed at a time, say x_2, keeping x_1 constant. To find out suitable direction,

$$\Phi(x_1, x_2, r) = \Phi(4086, 0.9, 1) = 1198.7949$$

and, $\Phi(4086, 1.1, 1) = 1198.5514$

After selecting the positive direction, a fixed step length of 0.5 is used and the valus are shown in Table 3.2.

Table 3.2: Values of Φ with variation of x_2

x_2	$\Phi(4086, x_2, 1)$
1.5	1198.1553
2	1197.8423
2.5	1197.7636
3	1198.0193

Now $\Phi(4086, 2.5, 1) = 1197.7636$

In order to vary x_1,

$\Phi(4087, 2.5, 1) = 1197.7634$

and $\Phi(4085, 2.5, 1) = 1197.7638$

Positive direction is selected and the univariate search is as shown in Table 3.3.

Table 3.3: Values of Φ with variation of x_1

x_1	$\Phi(x_1, 2.5, 1)$
4087	1197.7634
4088	1197.7634
4089	1197.7633
4090	1197.7634

At the end of cycle 1, $\Phi(4089, 2.5, 1) = 1197.7633$

Cycle 2:

For the variation of x_2,

$\Phi(4089, 2.4,1) = 1197.7574$

$\Phi(4089,2.6,1) = 1197.7819$

Negative direction is selected, and values are

x_2:	2.4	2.3
Φ:	1197.7574	1197.7632

Now

$\Phi(4089, 2.4, 1) = 1197.7574$

For the variation of x_1, no improvement is there in both the directions. The search is completed with respect to $r = 1$.

As the multiplication factor = 0.01,

next value of $r = 1 \times 0.01 = 0.01$

***r* = 0.01:**

Cycle 1:

Using $r = 0.01$, from (3.10), $\Phi(4089, 2.4, 0.01) = 1197.1387$.

Function value, $f(x_1, x_2)$ is obtained as equal to 1197.1324

In order to minimize Φ, for the variation of x_2, positive direction is chosen and the values are:

x_2:	2.5	2.6	2.7	2.8	2.9
Φ:	1197.0967	1197.0747	1197.0529	1197.0379	1197.0297
x_2:	3.0	3.1			
Φ:	1197.0284	1197.0339			

Now

$\Phi(4089, 3, 0.01) = 1197.0284$

Positive direction is selected for variation of x_1 and the values are:

x_1:	4090	4091	4092	4093
Φ:	1197.0282	1197.0281	1197.028	1197.0281

At the end of cycle 1, $\Phi(4092, 3, 0.01) = 1197.028$

Cycle 2:

For the variation of x_2

$\Phi(4092, 3.1, 0.01) = 1197.033$

and $\Phi(4092, 2.9, 0.01) = 1197.029$

No improvement is possible in either direction, and with respect to $r = 0.01$,

$$x_1 = 4092 \text{ units}$$

$$x_2 = 3 \text{ units}$$

$$f(x_1, x_2) = \text{Rs. } 1197.02$$

and $$\Phi(x_1, x_2, r) = 1197.028$$

Till now, this procedure is summarized in Table 3.4. The process may be continued to any desired accuracy with next value of r = (previous value of r) × 0.01 = 0.01 × 0.01 = 0.0001.

Table 3.4: Summary of the results

Value of r	*Starting point*	*Number of cycles to minimize by univariate method*	*Optimum x_1 and x_2*	*Φ**	*f**
1	$x_1 = 4086$ $x_2 = 1$	2	$x_1 = 4089$ $x_2 = 2.4$	1197.757	1197.13
0.01	$x_1 = 4089$ $x_2 = 2.4$	1	$x_1 = 4092$ $x_2 = 3$	1197.028	1197.02

4

GEOMETRIC PROGRAMMING

Geometric programming is suitable for a specific case of nonlinear objective function, namely, posynomial function.

4.1 INTRODUCTION

In some of the applications, either open or closed containers are fabricated from the sheet metal, which are used for transporting certain volume of goods from one place to another. Cross-section of the container may be rectangular, square or circular. A cylindrical vessel or a container with circular cross-section is shown in Fig. 4.1.

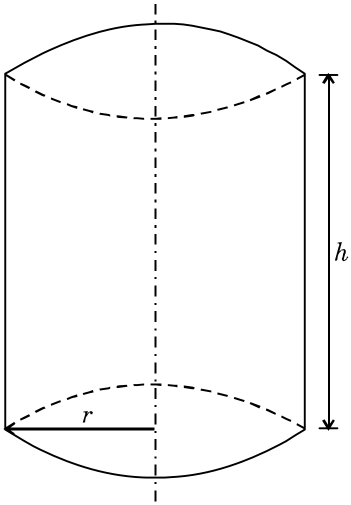


Fig. 4.1 A cylindrical box.

Assuming it an open box (*i.e.* without a lid), area of the sheet metal used for bottom = πr^2

where r = Radius in m.

Considering cost of fabricating including material cost as Rs. 200 per m^2,

$$\text{the relevant cost for the bottom} = 200\pi r^2 \qquad ...(4.1)$$

$$\text{Area of the vertical portion} = 2\pi rh$$

where h = height in m.

Considering cost per m^2 as Rs. 230,

$$\text{the relevant cost} = 230 \times 2\pi rh = 460\pi rh \qquad ...(4.2)$$

If 90 m^3 of the goods are to be transported, then number of trips needed = $\dfrac{90}{\pi r^2 h}$

Considering the transportation cost as Rs. 4 for each trip (to and fro),

$$\text{the cost of transportation} = 4 \times \frac{90}{\pi r^2 h} = \frac{360 r^{-2} h^{-1}}{\pi} \qquad ...(4.3)$$

Adding equations (4.1) (4.2) and (4.3), the total relevant cost function,

$$f(r, h) = 200\pi r^2 + 460\pi rh + \frac{360}{\pi} r^{-2} h^{-1} \qquad ...(4.4)$$

Refer to equation (4.1), the coefficient is 200 π and exponent of the variable r is 2, and exponent of h is zero, because it can be written as $200\pi r^2 h^0$

Refer to equation (4.4), the coefficient are 200π, 460π and 360/π, which are all positive. Exponents of variables r and h are real *i.e.,* either positive or negative or zero. Further r and h are positive variables. Such a function f represented by (4.4) is a posynomial. A posynomial function has the following characteristics:

(*i*) All the variables are positive.

(*ii*) Real exponents for these variables.

(*iii*) Positive coefficient in each component of the O.F.

Posynomial function, f, stated by (4.4) has three cost components :

(i) $M_1 = 200\pi r^2 h^0$

(ii) $M_2 = 460\pi r^1 h^1$

(iii) $M_3 = \frac{360}{\pi} r^{-2} h^{-1}$

and $$f = M_1 + M_2 + M_3 = \sum_{i=1}^{N} M_i$$

where N = number of components in the O.F. = 3.

Number of variables in the present example, $n = 2$. Grametric programming is suitable for minimizing a posynomial function, as explained above.

4.2 MATHEMATICAL ANALYSIS

A posynomial function can be written as,

$$f = \sum_{i=1}^{N} M_i = \sum_{i=1}^{N} C_i \cdot x_1^{a_{1i}} \cdot x_2^{a_{2i}} \ldots . x_n^{a_{ni}} \qquad \ldots(4.5)$$

where C_i = Positive coefficient

$a_{1i}, a_{2i}, \ldots\ldots., a_{ni}$ = Real exponents

$x_1, x_2, \ldots\ldots., x_n$ = Positive variables

n = Number of variables in the problem

N = Number of components in the O.F.

Differentiating equation (4.5) partially with respect to x_1 and equating to zero,

$$\frac{\partial f}{\partial x_1} = \sum_{i=1}^{N} C_i . a_{1i} . x_1^{a_{1i}-1} . x_2^{a2i} \ldots x_n^{a_{ni}} = \frac{1}{x_1} \sum_{i=1}^{N} a_{1i} . M_i = 0$$

Similarly $$\frac{\partial f}{\partial x_2} = \frac{1}{x_2} \sum_{i=1}^{N} a_{2i} . M_i = 0$$

$$\frac{\partial f}{\partial x_n} = \frac{1}{x_n} \sum_{i=1}^{N} a_{ni} \cdot M_i = 0$$

To generalize,

$$\frac{\partial f}{\partial x_k} = \frac{1}{x_k} \sum_{i=1}^{N} a_{ki} \cdot M_i = 0, k = 1, 2, \ldots n \quad \ldots(4.6)$$

Let $x_1^*, x_2^*, \ldots x_n^*$ be the optional values of the variables corresponding to minimization of equation (4.5).

$M_1^*, M_2^*, \ldots M_N^*$ are the values after substituting the optimum variables in each component of the O.F.

From (4.5), optimum value of the O.F.,

$$f^* = \sum_{i=1}^{N} M_i^* = M_1^* + M_2^* + \ldots + M_N^*$$

Dividing by f^* on both sides,

$$\frac{M_1^*}{f^*} + \frac{M_2^*}{f^*} + \ldots + \frac{M_N^*}{f^*} = \frac{f^*}{f^*} = 1$$

or $w_1 + w_2 + \ldots + w_N = 1$

where $w_1, w_2, \ldots, w_N$ are the positive fractions indicating the relative contribution of each optimal component M_1^*, M_2, ..., M_N^* respectively to the minimum f^*,

To generalize, $\sum_{i=1}^{N} w_i = 1$ ...(4.7)

where $$w_i = \frac{M_i^*}{f^*} \quad \ldots(4.8)$$

or $$M_i^* = \omega_i \cdot f^* \quad \ldots(4.9)$$

Equation (4.6) can be written as,

$\sum_{i=1}^{N} a_{ki} \cdot M_i^* = 0$, substituting the optimum values and multiplying with $x_k{}^*$ on both sides.

Putting the value of $M_i{}^*$ from (4.9) in above equation,

$$f^* \sum_{i=1}^{N} a_{ki} . w_i = 0$$

or
$$\sum_{i=1}^{N} a_{ki} . w_i = 0, \; k = 1, 2, ..., n \qquad ...(4.10)$$

The necessary conditions obtained by (4.7) and (4.10) are used to solve the problem. A unique solution is obtained for w_i, i = 1, 2,.... N if N = n + 1, as will be observed later. This type of problem is having zero degree of difficulty as N – n – 1 = 0, where degree of difficulty = (N – n – 1). A comparatively complicated problem may occur if N > (n + 1) because degree of difficulty $\geq$ 1.

Now,
$$f^* = (f^*)^{\sum_{i=1}^{N} w_i}, \text{ using equation (4.7)}$$

$$= (f^*)^{w_1} (f^*)^{w_2} \ldots (f^*)^{w_N}$$

$$= \left(\frac{M_1^*}{w_1}\right)^{w_1} \left(\frac{M_2^*}{w_2}\right)^{w_2} \ldots \left(\frac{M_N^*}{w_N}\right)^{w_N},$$

since from equation (4.8),

$$f^* = \frac{M_i^*}{w_i} \text{ or } f^* = \frac{M_1^*}{w_1} = \frac{M_2^*}{w_2} = \ldots = \frac{M_N^*}{w_N}$$

As $M_i{}^* = c_i . x_1{}^{*a_{1i}} . x_2{}^{*a_{2i}} \ldots x_2{}^{*a_{ni}}$, $\quad i = 1, 2, \ldots, N$ ...(4.11)

$$f^* = \left[\frac{C_1}{w_1} . x_1{}^{*a_{11}} . x_2{}^{*a_{21}} \ldots x_n{}^{*a_{n1}} \right]^{w_1}$$

$$\left[\frac{C_2}{w_2} . x_1{}^{*a_{12}} . x_2{}^{*a_{22}} \ldots x_n{}^{*a_{n2}} \right]^{w_2} \ldots$$

$$\left[\frac{C_N}{w_N} . x_1^{*a_{1N}} . x_2^{*a_{2N}} \ldots x_n^{*a_{nN}}\right]^{w_N}$$

$$\text{or} \quad f^* = \left[\frac{C_1}{w_1}\right]^{w_1}\left[\frac{C_2}{w_2}\right]^{w_2}\ldots\left[\frac{C_N}{w_N}\right]^{w_N}\left[(x_1^*)^{a_{11}w_1 + a_{12}w_2 + \ldots a_{1N}w_N}\right.$$

$$\left.(x_2^*)^{a_{21}w_1 + a_{22}w_2 + \ldots + a_{2N}w_N} \ldots (x_n^*)^{a_{n1}w_1 + a_{n2}w_2 + \ldots + a_{nN}w_N}\right]$$

$$\text{or } f^* = \left[\frac{C_1}{w_1}\right]^{w_1}\left[\frac{C_2}{w_2}\right]^{w_2}\ldots\left[\frac{C_N}{w_N}\right]^{w_N}\left[(x_1^*)^{\sum_{i=1}^{N} a_{1i}w_i} . (x_2^*)^{\sum_{i=1}^{N} a_{2i}w_i} \ldots (x_n^*)^{\sum_{i=1}^{N} a_{ni}w_i}\right]$$

Using Equation (4.10),

$$f^* = \left[\frac{C_1}{w_1}\right]^{w_1}\left[\frac{C_2}{w_2}\right]^{w_2}\ldots\left[\frac{C_N}{w_N}\right]^{w_N}\left[(x_1^*)^0 . (x_2^*)^0 \ldots (x_n^*)^0\right]$$

$$\text{or} \quad f^* = \left[\frac{C_1}{w_1}\right]^{w_1}\left[\frac{C_2}{w_2}\right]^{w_2}\ldots\left[\frac{C_N}{w_N}\right]^{w_N} \qquad \ldots(4.12)$$

In order to minimize any posynomial function, w_1, w_2, ... w_N are obtained using equation (4.7) and (4.10). These are substituted in equation (4.12) to get optimum function value f^*. Then from equation (4.9), M_1^*, M_2^*, ..., M_N^* are evaluated and finally using (4.11), optimum x_1^*, x_2^*, ...x_n^* *are* computed.

In most of the other methods, optimum variables are obtained and then by substituting them in the O.F., optimum function value is computed. While in the geometric programming, the reverse takes place. After geting the optimum function value, variables are evaluated.

4.3 EXAMPLES

As discussed before, the geometric programming can be effectively applied it any problem can be converted to a posynomial function having positive coefficient in each component, positive variables and their real exponents. In this section, the method is illustrated by some examples.

Example 4.1. *A manufacturing industry produces its finished product in batches. From the supply chain management point of view, it wants to incorporate raw material ordering policy in its total annual relevant cost and further to minimise the costs. The management has estimated the total cost function f, as follows :*

$$f = 0.15 x_1 x_2^{1.1} + \frac{63158\, x_2}{x_1} + \frac{5.75\, x_1^{0.6}}{x_2}$$

where x_1 *= Manufacturing batch size*

and x_2 *= Frequency of ordering of raw material in a manufacturing cycle.*

Minimize f using geometric programming.

Solution.

Now $f = 0.15\, x_1^1\, x_2^{1.1} + 63158\, x_1^{-1}\, x_2^1 + 5.75\, x_1^{0.6}\, x_2^{-1}$

Number of components in the O.F., N = 3

Number of variables in the problem, $n = 2$

From (4.7), $w_1 + w_2 + w_3 = 1$ ...(4.13)

From (4.10), $\sum_{i=1}^{3} a_{ki} . w_i = 0,\ k = 1, 2$

For $k = 1$, $\sum_{i=1}^{3} a_{1i} . w_i = 0$, or $a_{11} w_1 + a_{12} w_2 + a_{13} w_3 = 0$

...(4.14)

For $k = 2$, $a_{21} w_1 + a_{22} w_2 + a_{23} w_3 = 0$ ...(4.15)

Equations (4.13), (4.14) and (4.15) may also be written in the matrix form as follows :

$$\begin{bmatrix} 1 & 1 & 1 \\ a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad ...(4.16)$$

where a_{11}, a_{12} and a_{13} are the exponent of x_1 in each component of the O.F. Similarly a_{21}, a_{22} and a_{23} are the exponent of x_2 in each component.

Now exponents of x_1 are 1, –1 and 0.6 in first, second and third component respectively. Similarly exponents of x_2 are 1.1, 1 and –1 respectively in the first, second and third component of the function *f*.

Using (4.13), (4.14), and (4.15), or a set of equations represented by (4.16),

$$w_1 + w_2 + w_3 = 1 \quad \text{...(4.17)}$$

$$w_1 - w_2 + 0.6\ w_3 = 0 \quad \text{...(4.18)}$$

$$1.1\ w_1 + w_2 - w_3 = 0 \quad \text{...(4.19)}$$

as
$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0.6 \\ 1.1 & 1 & -1 \end{bmatrix}$$

Solving (4.17), (4.18) and (4.19),

$w_1 = 0.096$, $w_2 = 0.4$, and $w_3 = 0.504$

From (4.12), $$f^* = \left[\frac{C_1}{w_1}\right]^{w_1} \left[\frac{C_2}{w_2}\right]^{w_2} \left[\frac{C_3}{w_3}\right]^{w_3} \text{ as N = 3}$$

As C_1, C_2 and C_3 are coefficients of each component in the O.F., *i.e.*, $C_1 = 0.15$, $C_2 = 63158$, and $C_3 = 5.75$, substituting the relevant values,

$$f^* = \left[\frac{0.15}{0.096}\right]^{0.096} \left[\frac{63158}{0.4}\right]^{0.4} \left[\frac{5.75}{0.504}\right]^{0.504}$$

$$= 427.37$$

From (4.9), $M_i^* = w_i.f^*$

$M_1^* = 0.15\ x_1^*,\ x_2^{*1.1} = w_1.f^* = 41.03$

or $$0.15 x_1^*\ x_2^{*1.1} = 41.03 \quad \text{....(4.20)}$$

Now $$M_2^* = \frac{63158\ x_2^*}{x_1^*} = w_2.f^*$$

or $$\frac{63158\, x_2^*}{x_1^*} = 0.4 \times 427.37 = 170.95 \qquad ...(4.21)$$

Similarly $$M_3^* = \frac{5.75\, x_1^{*0.6}}{x_2^*} = w_3 . f^* = 215.39 \qquad ...(4.22)$$

On solving equations (4.20), (4.21) and (4.22), following approximate optimal values are obtained,

$$x_1{}^* = 320.17$$

and $$x_2{}^* = 0.87$$

Example 4.2. *In section 4.1, a problem concerning fabrication of a cylindrical box is discussed with the objective of minimizing total relevant cost, which is the sum of transportation and fabrication cost. The O.F. is formulated and given by equation (4.4). This is as follows-*

$$f\,(r,\, h) = 200\pi r^2 + 460\pi rh + \frac{360}{\pi} r^{-2} h^{-1}$$

Apply the geometric programming.

Solution.

Now $$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -2 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

or $$w_1 + w_2 + w_3 = 1$$

$$2w_1 + w_2 - 2w_3 = 0$$

$$w_2 - w_3 = 0$$

On solving, $w_1 = 0.2$

$$w_2 = 0.4$$

$$w_3 = 0.4$$

Further the coefficients of the components of the O.F. are, $C_1 = 200\pi$, $C_2 = 460\pi$, $C_3 = \frac{360}{\pi}$, and

$$f^* = \left[\frac{C_1}{w_1}\right]^{w_1} \left[\frac{C_2}{w_2}\right]^{w_2} \left[\frac{C_3}{w_3}\right]^{w_3}$$

Substituting the values, $f^* = 1274.69$

From (4.9), each component $M_i^* = w_i f^*$, and therefore

$$M_1^* = 200\pi r^2 = w_1 f^* = 0.2 \times 1274.69$$

or $$200\pi r^2 = 254.938$$

or $$r = 0.637$$

$$M_2^* = 460\pi rh = w_2 . f^* = 0.4 \times 1274.69$$

or $$rh = 0.3528$$

As $$r = 0.637, \ h = 0.554$$

The optimum results are obtained as follows

$$r = 0.637 \text{ m}$$

$$h = 0.554 \text{ m}$$

and $$f^* = \text{Rs. } 1274.69$$

Example 4.3. *An industry wants to fabricate an open box of mild steel as shown in Fig. 4.2. This box of rectangular cross-section will be used for transporting certain volume of small items for a project. This volume is estimated to be 90 m^3. Transportation cost for each trip (to and fro) is Rs. 6. Assume that the thickness of mild steel sheet used is 2 mm. Cost of the material and fabrication costs are estimated to be Rs. 30 per kg. Find out the dimensions of the box in order to minimize the total relevant cost. Consider the weight of the material as 7.85 gm/cm^3.*

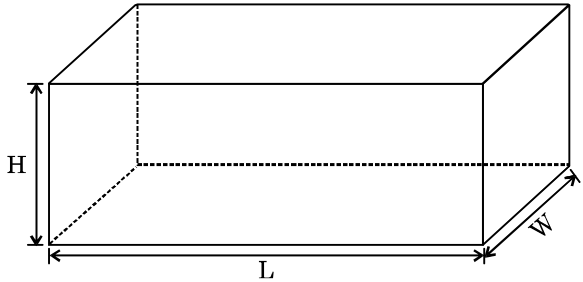


Fig. 4.2 An open box of rectangular cross-section.

Solution.

Let L = Length of the box in cm

W = Width of the box in cm

and $\quad H$ = Height of the box in cm

Now, 2 Nos. of steel sheets used are of the area (L × H) cm^2

2 Nos. of sheets are having area (W × H) cm^2 each

1 No. of sheet with area (L × W) cm^2

As the thickness of the material is 2 mm = 0.2 cm,

Volume of the material = 0.2 [2(L × H) + 2 (W × H) + (L × W)] cm^3

and $\quad$ weight = 7.85 × 0.2 [2LH + 2WH + LW) gm

$= 1.57 \times 10^{-3}$ [2LH + 2WH + LW) kg

Cost of the material and fabrication cost in Rs.,

$= 30 \times 1.57 \times 10^{-3}$ [2LH + 2WH + LW)

= 0.0942 LH + 0.0942 WH + 0.0471 LW

As the number of trips needed = $\dfrac{90 \times 10^6}{LWH}$,

Cost of transportation = $\dfrac{6 \times 90 \times 10^6}{LWH} = \dfrac{540 \times 10^6}{LWH}$

Total relevant cost function,

$f = 0.0942\ LH + 0.0942\ WH + 0.0471\ LW + 540 \times 10^6\ L^{-1}\ W^{-1}\ H^{-1}$

Number of cost components in f are 4.

	Component			
	1	2	3	4
Exponents of L	1	0	1	–1
Exponents of W	0	1	1	–1
Exponents of H	1	1	0	–1

and therefore,

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & -1 \\ 1 & 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{or} \quad w_1 + w_2 + w_3 + w_4 = 1$$
$$w_1 + w_3 - w_4 = 0$$
$$w_2 + w_3 - w_4 = 0$$
$$w_1 + w_2 - w_4 = 0$$

On solving,

$$w_1 = w_2 = w_3 = 0.2$$

and $w_4 = 0.4$

Now $C_1 = 0.0942$

$$C_2 = 0.0942$$
$$C_3 = 0.0471$$

and $C_4 = 540 \times 10^6$

$$f^* = \left[\frac{C_1}{w_1}\right]^{w_1} \left[\frac{C_2}{w_2}\right]^{w_2} \left[\frac{C_3}{w_3}\right]^{w_3} \left[\frac{C_4}{w_4}\right]^{w_4}$$

$$= \left[\frac{0.0942}{0.2}\right]^{0.2} \left[\frac{0.0942}{0.2}\right]^{0.2} \left[\frac{0.0471}{0.2}\right]^{0.2} \left[\frac{540 \times 10^6}{0.4}\right]^{0.4}$$

$$= 2487.37$$

Cost component $M_i^* = w_i . f^*, \; i = 1, 2, 3, 4$

$$M_1^* = 0.0942 \text{ LH} = 0.2 \times 2487.37$$

or $\text{LH} = 5281.04$ ...(4.23)

$$M_2^* = 0.0942 \text{ WH} = w_2 . f^* = 0.2 \times 2487.37$$

or $\text{WH} = 5281.04$ ...(4.24)

$$M_3^* = 0.0471 \text{ LW} = 0.2 \times 2487.37$$

or $\text{LW} = 10562.08$ ...(4.25)

$$M_4^* = \frac{540 \times 10^6}{\text{LWH}} = w_4 f^* = 0.4 \times 2487.37$$

or $\text{LWH} = 542741.93$ ...(4.26)

Substituting equation (4.25),

$$\text{H} = \frac{542741.93}{10562.08} = 51.39 \text{ cm}$$

From (4.24), $W = 102.77$ cm

From (4.23), $L = 102.77$ cm

Therefore the optimal results are as follows :

$$L^* = 102.77 \text{ cm}$$

$$W^* = 102.77 \text{ cm}$$

$$H^* = 51.39 \text{ cm}$$

and $$f^* = \text{Rs. } 2487.37$$

5

MULTI-VARIABLE OPTIMIZATION

Convex and concave functions were discussed in Chapter 1 in the context of a single variable problem. For the multi-variable problem also, optimum results can be achieved by differentiating the O.F. partially with respect to each variable and equating to zero, if it can be ascertained whether the function is convex or concave.

5.1 CONVEX/CONCAVE FUNCTION

The first step to determine convexity/concavity, is to construct the Hessian. Hessian is a matrix, the elements of which are $\frac{\partial^2 f}{\partial x_i \, \partial x_j}$, where i and j are row and column respectively corresponding to that element.

Example 5.1. *Construct the Hessian matrix for—*

(a) two variable problem

(b) three variable problem

(c) Four variable problem.

Solution. (a) The Hessian matrix J for a two variable problem is as follows:

$$(a) \qquad J = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} \\ \frac{\partial^2 f}{\partial x_2 \, \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix}$$

(*b*) $$J = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} \\ \frac{\partial^2 f}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} \end{bmatrix}$$

(*c*) $$J = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \frac{\partial^2 f}{\partial x_1 \partial x_2} & \frac{\partial^2 f}{\partial x_1 \partial x_3} & \frac{\partial^2 f}{\partial x_1 \partial x_4} \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} & \frac{\partial^2 f}{\partial x_2 \partial x_3} & \frac{\partial^2 f}{\partial x_2 \partial x_4} \\ \frac{\partial^2 f}{\partial x_3 \partial x_1} & \frac{\partial^2 f}{\partial x_3 \partial x_2} & \frac{\partial^2 f}{\partial x_3^2} & \frac{\partial^2 f}{\partial x_3 \partial x_4} \\ \frac{\partial^2 f}{\partial x_4 \partial x_1} & \frac{\partial^2 f}{\partial x_4 \partial x_2} & \frac{\partial^2 f}{\partial x_4 \partial x_3} & \frac{\partial^2 f}{\partial x_4^2} \end{bmatrix}$$

5.1.1 Convex Function

In order to ascertain a convex function, the principal minors of the Hessian matrix should have positive sign. Let a Hessian be $(n \times n)$ matrix. An mth leading principal minor is the determinant of the matrix obtained after deleting the last $(n - m)$ rows and corresponding columns.

Example 5.2. *Ascertain whether the following function is convex for positive values of* x_1 *and* x_2 : $f(x_1, x_2) = 5x_1^{-1} + 9x_1 - 10x_2 + 4x_2^2$.

Solution. Now $$\frac{\partial f}{\partial x_1} = 9 - \frac{5}{x_1^2}$$

$$\frac{\partial^2 f}{\partial x_2 \partial x_1} = \frac{\partial^2 f}{\partial x_1 \partial x_2} = 0$$

$$\frac{\partial^2 f}{\partial x_1^2} = \frac{10}{x_1^3}$$

$$\frac{\partial^2 f}{\partial x_2^2} = \frac{\partial}{\partial x_2}\left(-10 + 8x_2\right) = 8$$

$$J = \begin{bmatrix} \frac{10}{x_1^3} & 0 \\ 0 & 8 \end{bmatrix}$$

When $m = 2$, $n - m = 2 - 2 = 0$ and no row or column are to be deleted and,

$$\text{second principal minor} = \begin{vmatrix} \frac{10}{x_1^3} & 0 \\ 0 & 8 \end{vmatrix}$$

$$= \frac{80}{x_1^3} - 0 = \frac{80}{x_1^3}$$

For the first principal minor, $m = 1$ and $n - m = 2 - 1 = 1$ and therefore last row and corresponding column are to be deleted.

After deleting row 2 and column 2, remaining value $= \frac{10}{x_1^3}$. First principal minor is $\frac{10}{x_1^3}$, which is positive. Second principal minor *i.e.,* $\frac{80}{x_1^3}$ is also positive for positive x_1. The given function is convex.

Example 5.3. *A function consisting of three variables x_1, x_2 and x_3 is minimized by equating partial derivatives of function with each variable and equating to zero. In order to ascertain the optimality with respect to the results achieved, develop the necessary conditions.*

Solution. Hessian matrix $= \begin{bmatrix} A & H & G \\ H & B & F \\ G & F & C \end{bmatrix}$

If $\quad A = \frac{\partial^2 f}{\partial x_1^2}, \quad B = \frac{\partial^2 f}{\partial x_2^2}, \quad C = \frac{\partial^2 f}{\partial x_3^2}$

$$H = \frac{\partial^2 f}{\partial x_1 \partial x_2}, \quad F = \frac{\partial^2 f}{\partial x_2 \partial x_3}, \quad \text{and} \quad G = \frac{\partial^2 f}{\partial x_3 \partial x_1}$$

First principal minor is obtained by deleting last two rows and last two columns, and the first condition is,

$$A > 0 \qquad ...(5.1)$$

Second principal minor is obtained by deleting last row and last column, and

$$\begin{vmatrix} A & H \\ H & B \end{vmatrix} > 0$$

or $$AB > H^2 \qquad ...(5.2)$$

Third principal minor should be greater than zero, *i.e.*

$$\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} > 0$$

or $$A \begin{vmatrix} B & F \\ F & C \end{vmatrix} - H \begin{vmatrix} H & F \\ G & C \end{vmatrix} + G \begin{vmatrix} H & B \\ G & F \end{vmatrix} > 0$$

or $$A(BC - F^2) - H(HC - FG) + G(HF - BG) > 0$$

or $$ABC + 2FGH > AF^2 + BG^2 + CH^2 \qquad ...(5.3)$$

In order to minimize f, optimum results are obtained by solving,

$$\frac{\partial f}{\partial x_1} = 0, \quad \frac{\partial f}{\partial x_2} = 0 \quad \text{and} \quad \frac{\partial f}{\partial x_3} = 0$$

The conditions (5.1), (5.2) and (5.3) will need to be satisfied for optimality.

5.1.2 Concave Function

As discussed before, mth principal minor is obtained as the determinant of the matrix or value achieved after deleting last $(n - m)$ rows and corresponding columns. In order to ascertain a concave function with respect to each variable, the principal minor should have the sign $(-1)^m$.

First principal minor should have $(-1)^1 = -1$ *i.e.,* negative sign, Second principal minor should have $(-1)^2$ = positive sign,

Third principal minor should have $(-1)^3$ = negative sign, and so on

Example 5.4. *Determine whether the following function is concave for positive variables—*

$$f(x_1, x_2) = 8x_2 - 7x_1 - 10x_1^{-1} - 6x_2^2$$

Solution.
$$\frac{\partial f}{\partial x_1} = -7 + \frac{10}{x_1^2}$$

$$\frac{\partial^2 f}{\partial x_1^2} = \frac{-20}{x_1^3}$$

$$\frac{\partial^2 f}{\partial x_1 \partial x_2} = \frac{\partial^2 f}{\partial x_2 \partial x_1} = 0$$

$$\frac{\partial^2 f}{\partial x_2^2} = -12$$

Now
$$J = \begin{bmatrix} -\frac{20}{x_1^3} & 0 \\ 0 & -12 \end{bmatrix}$$

First principal minor $= \frac{-20}{x_1^3}$

This is having negative sign, *i.e.,* less than zero.

Second principal minor $= \begin{bmatrix} -\frac{20}{x_1^3} & 0 \\ 0 & -12 \end{bmatrix}$

$$= \frac{240}{x_1^3}$$

which is having positive sign, *i.e.,* greater than zero.

Therefore the given function is concave with respect to x_1 and x_2 each. In order to maximize this function, optimum results can be obtained by solving,

$$\frac{\partial f}{\partial x_1} = 0 \text{ and } \frac{\partial f}{\partial x_2} = 0$$

Example 5.5. *A function of three variables is maximized by solving,*

$$\frac{\partial f}{\partial x_1} = 0, \quad \frac{\partial f}{\partial x_2} = 0 \text{ and } \frac{\partial f}{\partial x_3} = 0$$

Develop the necessary conditions in order to guarantee the optimality.

Solution. As mentioned in Example 5.2, let

$$A = \frac{\partial^2 f}{\partial x_1^2}, \qquad B = \frac{\partial^2 f}{\partial x_2^2}, \qquad C = \frac{\partial^2 f}{\partial x_3^2}$$

$$H = \frac{\partial^2 f}{\partial x_1 \partial x_2}, \quad F = \frac{\partial^2 f}{\partial x_2 \partial x_3}, \quad \text{and} \quad G = \frac{\partial^2 f}{\partial x_3 \partial x_1}$$

$$\text{Hessian matrix} = \begin{bmatrix} A & H & G \\ H & B & F \\ G & F & C \end{bmatrix}$$

odd principal minors should be less than zero, *i.e.,* have negative sign. Even principal minors should be greater than zero, *i.e.,* have positive sign.

First principal minor = A,

and $$A < 0 \qquad \text{...(5.4)}$$

Second principal minor,

$$\begin{vmatrix} A & H \\ H & B \end{vmatrix} > 0$$

or $$AB > H^2 \qquad \text{...(5.5)}$$

Third principal minor,

$$\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} < 0$$

or $$ABC + 2FGH < AF^2 + BG^2 + CH^2 \qquad \text{...(5.6)}$$

The conditions (5.4), (5.5) and (5.6) will need to be satisfied for optimality.

5.2 LAGRANGEAN METHOD

Consider the following multivariable problem with equality constraints.

Minimize $f(x_1, x_2, ..., x_n)$

subject to $h_i (x_1, x_2, ..., x_n) = 0, \ i = 1, 2, ... \ m$

The O.F. consists of n variables and m number of constraints, *i.e.*, $h_1, h_2, ..., h_m$ are imposed on it. The constraints are in the '=0' form. The method of Lagrange multipliers is useful in order to optimize such kind of problems. A new function, *i.e.*, Lagrange function, L is formed,

$$L(x_1, x_2, ..., x_n, \lambda_1, \lambda_2, ..., \lambda_m) = f(x_1, x_2, ..., x_n) + \sum_{i=1}^{m} \lambda_i h_i (x_1, x_2, ..., x_n) \quad ...(5.7)$$

where $\lambda_i, \ i = 1, 2, ..., m$ are called Lagrange multipliers. Negative sign may be used in equation (5.7) in a case of maximization problem.

Equation (5.7) is a function of $(n+m)$ variables including the Lagrange multipliers. Differentiating partially with respect to each variable and equating to zero, yields the optimal values after solving.

Example 5.6. *Minimize the following function using the method of Lagrange multipliers.*

$$f(x_1, x_2, x_3) = \frac{250}{x_1} + 1090x_1 - 7x_2 - 5x_3 + \frac{7x_2^2}{40x_1} + \frac{5x_3^2}{16x_1}$$

subject to $x_1 + x_3 = 4$

and $x_2 + x_3 = 12$

Solution. Both the constraints are to be written in the following form—

$$x_1 + x_3 - 4 = 0$$

and $$x_2 + x_3 - 12 = 0$$

Lagrange function

$$L(x_1, x_2, x_3, \lambda_1, \lambda_2) = \frac{250}{x_1} + 1090x_1 - 7x_2 - 5x_3 + \frac{7x_2^2}{40x_1} + \frac{5x_3^2}{16x_1} + \lambda_1 (x_1 + x_3 - 4) + \lambda_2 (x_2 + x_3 - 12)$$

Now,

$$\frac{\partial L}{\partial x_1} = \frac{-250}{x_1^2} + 1090 - \frac{7x_2^2}{40x_1^2} - \frac{5x_3^2}{16x_1^2} + \lambda_1 = 0 \quad ...(5.8)$$

$$\frac{\partial L}{\partial x_2} = -7 + \frac{14x_2}{40x_1} + \lambda_2 = 0 \quad ...(5.9)$$

$$\frac{\partial L}{\partial x_3} = -5 + \frac{10x_3}{16x_1} + \lambda_1 + \lambda_2 = 0 \quad ...(5.10)$$

$$\frac{\partial L}{\partial \lambda_1} = x_1 + x_3 - 4 = 0 \quad ...(5.11)$$

$$\frac{\partial L}{\partial \lambda_2} = x_2 + x_3 - 12 = 0 \quad ...(5.12)$$

Optimum values of all the five variables are obtained by solving these equations.

Subtracting (5.11) from (5.12),

$$x_2 = x_1 + 8 \quad ...(5.13)$$

From (5.11), $x_3 = 4 - x_1$ ...(5.14)

Using (5.9) and (5.10),

$$\lambda_2 = 7 - \frac{14x_2}{40x_1} = 5 - \frac{10x_3}{16x_1} - \lambda_1 \quad ...(5.15)$$

Substituting (5.13) and (5.14),

$$\lambda_1 = \frac{3}{10x_1} - \frac{41}{40} \quad ...(5.16)$$

Substituting (5.13), (5.14) and (5.16) in equation (5.8) and on solving,

$$x_1 = 0.49$$

From (5.16), $\lambda_1 = -0.41$

From (5.13), $x_2 = 8.49$

From (5.15), $\lambda_2 = 7 - \frac{14x_2}{40x_1} = 0.94$

From (5.14), $x_3 = 3.51$

Following optimal values are obtained,

$$x_1 = 0.49,\ x_2 = 8.49,\ x_3 = 3.51,$$

$$\lambda_1 = -0.41 \text{ and } \lambda_2 = 0.94$$

Substituting x_1, x_2 and x_3,

optimal function value, $f = 1000.92$

Example 5.7. *Minimize the following function using the Lagrangean method—*

$$f(x_1, x_2) = 6x_1 + \frac{96}{x_1} + \frac{4x_2}{x_1} + \frac{x_1}{x_2} \quad \text{...(5.17)}$$

subject to $x_1 + x_2 = 6$ ...(5.18)

Also discuss the physical significance of Lagrange multiplier.

Solution. The constraint is,

$$x_1 + x_2 - 6 = 0$$

and the Lagrange function,

$$L(x_1, x_2, \lambda) = 6x_1 + \frac{96}{x_1} + \frac{4x_2}{x_1} + \frac{x_1}{x_2} + \lambda(x_1 + x_2 - 6)$$

Now

$$\frac{\partial L}{\partial x_1} = 6 - \frac{96}{x_1^2} - \frac{4x_2}{x_1^2} + \frac{1}{x_2} + \lambda = 0 \quad \text{...(5.19)}$$

$$\frac{\partial L}{\partial x_2} = \frac{4}{x_1} - \frac{x_1}{x_2^2} + \lambda = 0 \quad \text{...(5.20)}$$

$$\frac{\partial L}{\partial \lambda} = x_1 + x_2 - 6 = 0 \quad \text{...(5.21)}$$

Solving these equations,

$$x_2 = 6 - x_1, \qquad \text{form (5.21)}$$

Substituting in (5.20), $\lambda = \dfrac{x_1}{(6 - x_1)^2} - \dfrac{4}{x_1}$

and from (5.19)

$$6 - \frac{96}{x_1^2} - \frac{4(6 - x_1)}{x_1^2} + \frac{1}{(6 - x_1)} + \frac{x_1}{(6 - x_1)^2} - \frac{4}{x_1} = 0$$

or

$$6 - \frac{120}{x_1^2} + \frac{1}{(6 - x_1)} + \frac{x_1}{(6 - x_1)^2} = 0$$

or $$6(6-x_1)^2 - \frac{120(6-x_1)^2}{x_1^2} + (6-x_1) + x_1 = 0$$

or $$(6-x_1)^2 - \frac{20(6-x_1)^2}{x_1^2} + 1 = 0$$

This is satisfied with $x_1 = 4$.

From (5.21), $x_2 = 2$ and from (5.20), $\lambda = 0$.

The optimal values of x_1, x_2 and λ are 4, 2 and 0 respectively. If the objective is to obtain the unconstrained minimum, then it can be observed that $x_1^* = 4$ and $x_2^* = 2$. In other words, the constraint (5.18) satisfies exactly the unconstrained optimal values and therefore the λ^* is obtained as zero.

optimal function value $f^* = f(x_1^*, x_2^*) = f(4, 2) = 52$

Let the R.H.S. of the constraint (5.18) be 7 in stead of 6, *i.e.,* the problem is to minimize.

$$f = 6x_1 + \frac{96}{x_1} + \frac{4x_2}{x_1} + \frac{x_1}{x_2}$$

subject to $x_1 + x_2 = 7$

Following the similar procedure, an equation in terms of x_1 is obtained as follows–

$$6 - \frac{124}{x_1^2} + \frac{1}{(7-x_1)} + \frac{x_1}{(7-x_1)^2} = 0$$

or $$6 - \frac{124}{x_1^2} + \frac{1}{(7-x_1)}\left[1 + \frac{x_1}{(7-x_1)}\right] = 0$$

or $$6 - \frac{124}{x_1^2} + \frac{7}{(7-x_1)^2} = 0$$

Above equation is satisfied with approximate value of x_1^* = 4.235. Remaining values are computed as $x_2^* = 2.765$

and $$\lambda^* = -0.39$$

optimum function value $f^* = 52.22$.

Relaxing the constraint further *i.e.,* $x_1 + x_2 = 8$, following approximate values are obtained,

$$x_1^* = 4.45,\ x_2^* = 3.55,\ \lambda^* = -0.5, \text{ and } f^* = 52.71.$$

With the negative value of the Lagrange multiplier λ, O.F. value increases as the constraint is relaxed.

If the constraint is tightened, say $x_1 + x_2 = 5$, then a positive value of λ is obtained. However, in this particular example, the O.F. value is observed to increase. This may be due to the O.F. being convex in terms of x_1 and x_2 each. The computational results are as follows—

$$x_1 = 3.65,\quad x_2 = 1.35$$
$$\lambda = 0.9,\quad f = 52.38$$

5.3 THE KUHN–TUCKER CONDITIONS

In the previous section, equality constraints were imposed on the O.F. But, in many cases, inequality constraints are used. For example, minimize

$$f = \frac{250}{x_1} + 1090x_1 - 7x_2 - 5x_3 + \frac{7x_2^2}{40x_1} + \frac{5x_3^2}{16x_1}$$

subject to $x_1 + x_3 - 4 \le 0$

$$x_2 + x_3 - 12 \le 0$$

The constraints may be converted into equality form as follows—

$$x_1 + x_3 - 4 + S_1^2 = 0$$

$$x_2 + x_3 - 12 + S_2^2 = 0$$

where S_1^2 and S_2^2 are slack variables and

$$S_1^2 \ge 0$$
$$S_2^2 \ge 0$$

Square of S_1 and S_2 are used in order to ensure the non-negative slack variables.

Lagrange function,

$$L = \frac{250}{x_1} + 1090x_1 - 7x_2 - 5x_3 + \frac{7x_2^2}{40x_1} + \frac{5x_3^2}{16x_1}$$
$$+ \lambda_1(x_1 + x_3 - 4 + S_1^2) + \lambda_2(x_2 + x_3 - 12 + S_2^2)$$

with $\lambda_1 \geq 0$

$\lambda_2 \geq 0$

The optimum solution is obtained by differentiating partially with respect to each variable and equating to zero. These variables are—

(*i*) Decision variables x_1, x_2, x_3

(ii) Lagrange multipliers λ_1, λ_2

(*iii*) Slack variables S_1, S_2

$$\frac{\partial L}{\partial S_1} = 2\lambda_1 S_1 = 0$$

or $\lambda_1 S_1 = 0$

Similarly, $\lambda_2 S_2 = 0$

In order to generalize, let the problem be—

Minimize $f(x_1, x_2, \ldots, x_n)$

subject to $h_i\,(x_1, x_2, \ldots, x_n) \leq 0$, $i = 1, 2, \ldots, m$

Now, using slack variables

$$h_i\,(x_1, x_2, \ldots, x_n) + S_i^2 = 0, \; i = 1, 2, \ldots, m$$

and $L\,(x_1, x_2, \ldots, x_n, \lambda_1, \lambda_2, \ldots, \lambda_m, S_1, S_2, \ldots, S_m)$

$$= f\,(x_1, x_2, \ldots, x_n) + \sum_{i=1}^{m} \lambda_i \left[h_i(x_1, x_2, \ldots, x_n) + S_i^2\right]$$

$$\frac{\partial L}{\partial x_j} = \frac{\partial f}{\partial x_j} + \sum_{i=1}^{m} \lambda_i \frac{\partial h_i}{\partial x_j}, \quad j = 1, 2, \ldots, n$$

Following conditions are known as the Kuhn-Tucker conditions—

$$\frac{\partial f}{\partial x_j} + \sum_{i=1}^{m} \lambda_i \frac{\partial h_i}{\partial x_j} = 0, \quad j = 1, 2, \ldots, n \qquad \ldots(5.22)$$

$$\lambda_i h_i = 0, \qquad i = 1, 2, \ldots, m \qquad \ldots(5.23)$$

$$h_i \leq 0, \qquad i = 1, 2, \ldots, m \qquad \ldots(5.24)$$

$$\lambda_i \geq 0, \qquad i = 1, 2, \ldots, m \qquad \ldots(5.25)$$

These conditions are needed to be satisfied for optimality. Equation (5.23) is obtained by analyzing the following conditions which were discussed before—

$$\lambda_i S_i = 0 \quad \text{...(5.26)}$$

and $$h_i + S_i^2 = 0 \quad \text{...(5.27)}$$

From (5.26), consider the following possibilities—

(*i*) $S_i = 0$ and $\lambda_i = 0$

Equation (5.23), *i.e.*, $\lambda_i h_i = 0$ is true as $\lambda_i = 0$

(*ii*) only $\lambda_i = 0$ and S_i(or S_i^2) > 0

$h_i \lambda_i = 0$ is satisfied

(*iii*) S_i (or S_i^2) = 0 and $\lambda_i \neq 0$

From (5.27), $h_i = 0$, and $h_i \lambda_i = 0$

Therefore (5.26) and (5.27), imply condition (5.23).

Example 5.8. *Minimize* $f = x_1 + \dfrac{4}{x_1} + \dfrac{x_2}{40x_1} + \dfrac{5x_1}{8x_2}$

subject to $x_1 + x_2 \leq 11$.

Using the Kuhn-Tucker conditions, obtain the optimal solution.

Solution. Now $h = x_1 + x_2 - 11 \leq 0$

and $$L = x_1 + \frac{4}{x_1} + \frac{x_2}{40x_1} + \frac{5x_1}{8x_2} + \lambda\,(x_1 + x_2 - 11)$$

Using the Kuhn-Tucker conditions from (5.22) to (5.25),

$$1 - \frac{4}{x_1^2} - \frac{x_2}{40x_1^2} + \frac{5}{8x_2} + \lambda = 0 \quad \text{...(5.28)}$$

$$\frac{1}{40x_1} - \frac{5x_1}{8x_2^2} + \lambda = 0 \quad \text{...(5.29)}$$

$$\lambda\,(x_1 + x_2 - 11) = 0 \quad \text{...(5.30)}$$

$$x_1 + x_2 - 11 \leq 0 \quad \text{...(5.31)}$$

$$\lambda \geq 0 \quad \text{...(5.32)}$$

From (5.30), three possibilities are as follows:

(*a*) $\lambda = 0$ and $x_1 + x_2 - 11 \neq 0$

(*b*) $\lambda = 0$ and $x_1 + x_2 - 11 = 0$

(*c*) $\lambda \neq 0$ and $x_1 + x_2 - 11 = 0$

Each of them are analyzed

(*a*) Substituting $\lambda = 0$ in (5.29),

$$\frac{1}{40x_1} - \frac{5x_1}{8x_2^2} = 0$$

or $$x_2 = 5x_1$$

Putting this and $\lambda = 0$ in (5.28),

$$x_1 = 2$$

and $$x_2 = 5x_1 = 10$$

As the condition (5.31) is not satisfied, this solution is not optimal.

(*b*) $\lambda = 0$ and $x_2 = 11 - x_1$

Substituting in (5.29),

$$x_1 = 11/6 \text{ and } x_2 = 11 - x_1 = 55/6$$

Using these values, as the (5.28) is not satisfied, it is not an optimal solution.

(*c*) $x_2 = 11 - x_1$

From (5.29), $$\lambda = \frac{5x_1}{8(11 - x_1)^2} - \frac{1}{40x_1} \quad ...(5.33)$$

From (5.28),

$$1 - \frac{4}{x_1^2} - \frac{(11 - x_1)}{40x_1^2} + \frac{5}{8(11 - x_1)} + \frac{5x_1}{8(11 - x_1)^2} - \frac{1}{40x_1} = 0$$

or $$40x_1^2 - 171 + \frac{275x_1^2}{(11 - x_1)^2} = 0$$

The equation is satisfied with $x_1^* = 1.9853$ and $x_2^* = 11 - x_1 = 9.0147$

From (5.33), $\lambda = 0.0027$

As all the conditions are satisfied, the optimal results are achieved with $f^* = 4.25$.

5.4 INDUSTRIAL ECONOMICS APPLICATION

Industrial economics relate to the managerial problems concerning cost-volume-profit analysis etc. In the context of nonlinear analysis, following functions (and their types) interact with each other—

(*i*) Nonlinear total cost function and linear total revenue function (Fig. 5.1).

(*ii*) Linear total cost function and nonlinear total revenue function (Fig. 5.2).

(*iii*) Nonlinear total cost function and nonlinear total revenue function (Fig. 5.3).

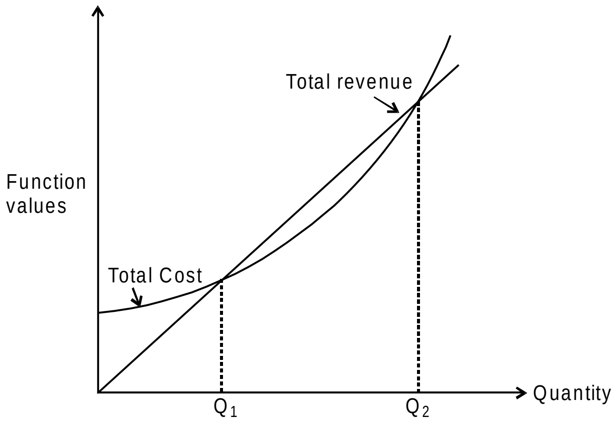


Fig. 5.1: Nonlinear total cost and linear revenue function.

As the profit is difference between revenue and cost, it is zero at the intersection of these functions. As shown in Fig. 5.1, Q_1 and Q_2 are the quantity corresponding to which the profit is zero. Maximum profit is obtained somewhere in the range $[Q_1, Q_2]$.

Similarly linear cost and nonlinear revenue function are shown in Fig. 5.2.

In some of the cases, both the functions may be nonlinear. This situation is represented by Fig. 5.3. Profit function is determined and then by equating it to zero, two breakeven points are obtained. Profit may also be maximized in the computed range of quantity.

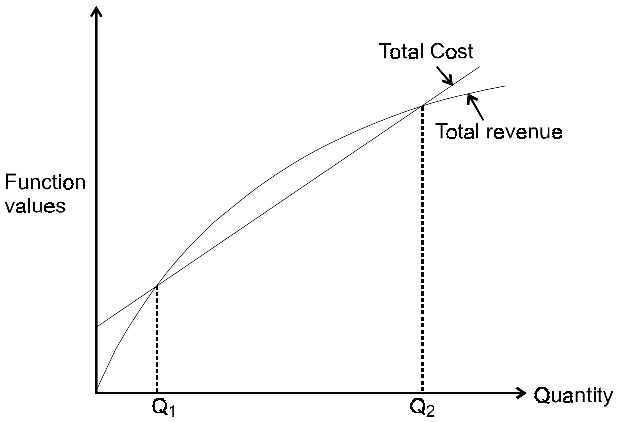


Fig. 5.2 : Linear total cost and nonlinear revenue function.

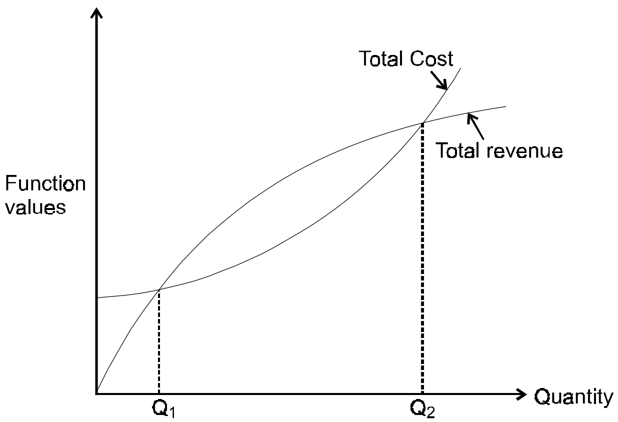


Fig. 5.3 : Nonlinear cost and nonlinear revenue function.

Example 5.9: *Consider the following functions using quantity Q,*

revenue function, $R = 25Q$

and total cost function, $C = 45 + Q + 2Q^2$

obtain the two values of breakeven points and maximize the profit in that range.

Solution. This case refers to the linear revenue function and nonlinear total cost function.

Profit, $P = R - C$

or $P = 25Q - [45 + Q + 2Q^2]$

or $$P = 24Q - 45 - 2Q^2$$

At the breakeven points, P = 0, and therefore

$24Q - 45 - 2Q^2 = 0$

or $2Q^2 - 24Q + 45 = 0$

or $$Q = \frac{24 \pm \sqrt{24^2 - 360}}{4}$$

$$= \frac{24 \pm \sqrt{216}}{4}$$

$\therefore$ $Q = 2.33$ or 9.67

Two values of breakeven points are 2.33 and 9.67 units. In order to maximize the profit,

$$\frac{dP}{dQ} = 0$$

or $24 - 4Q = 0$

or optimum $Q = 6$ units

optimality may be ensured by getting negative second order derivative.

Example 5.10. *Let, the revenue function, $R = 55Q - 3Q^2$ and total cost function, $C = 30 + 25Q$.*

Solve the problem by obtaining breakeven points and maximizing the profit.

Solution. The present case refers to the linear total cost and nonlinear revenue function.

Now profit $P = R - C = 30Q - 3Q^2 - 30$

To compute the breakeven points,

$30Q - 3Q^2 - 30 = 0$

or $3Q^2 - 30Q + 30 = 0$

or $$Q = \frac{30 \pm \sqrt{540}}{6}$$

The breakeven points are 1.13 and 8.87.

To maximize the profit,

$$\frac{dP}{dQ} = 0$$

or $30 - 6Q = 0$

or optimum $Q = 5$ units.

Example 5.11. *Calculate the quantity at which there is no profit– no loss and the quantity corresponding to maximum profit for,*

revenue function, $R = 71Q - 3Q^2$

and cost function, $C = 80 + Q + 2Q^2$

Solution. In this example, the revenue as well as cost functions are nonlinear.

Profit function, $P = 71Q - 3Q^2 - [80 + Q + 2Q^2]$

$= 70Q - 5Q^2 - 80$

At $P = 0, \quad 5Q^2 - 70Q + 80 = 0$

and $$Q = \frac{70 \pm \sqrt{3300}}{10}$$

The quantity at which there is no profit no loss, are

$Q = 1.25$ and 12.74 units.

For the maximum profit,

$$\frac{dP}{dQ} = 70 - 10Q = 0$$

and the quantity corresponding to maximum profit is equal to 7 units.

A multivariable problem occurs when profit function consists of more variables related to multiple items.

5.5 INVENTORY APPLICATION

As shown in Fig. 5.4, quantity Q is ordered periodically by a business firm. The firm is also allowing maximum shortage quantity J, which is backordered completely.

Let,

D = Annual demand or demand rate per year

K = Annual backordering or shortage cost per unit

I = Annual inventory holding cost per unit

C = Fixed ordering cost

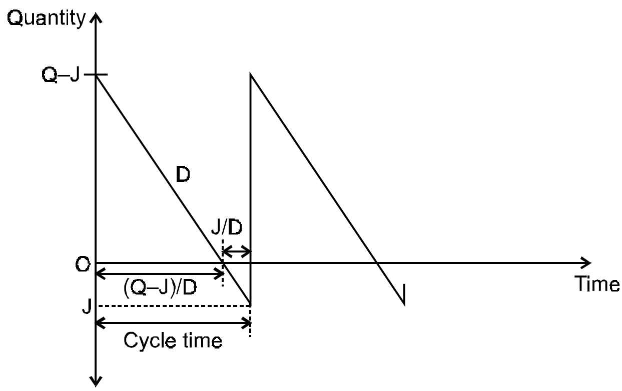


Fig. 5.4 : Inventory cycle

Quantity Q is being ordered and it is received instantaneously. Then it is consumed at a rate D units per year. As the shortages are allowed in the system, negative quantities are represented in Fig. 5.4. Similar cycle will start repeating when the maximum allowable shortages are J.

The cycle time consists of two components—

(*i*) (Q – J)/D, *i.e.,* the time in which quantity (Q – J) is consumed completely at the rate D, and

(*ii*) J/D,

$$\text{Cycle time} = \frac{(Q-J)}{D} + \frac{J}{D}$$

$$= \frac{Q}{D} \qquad \text{...(5.34)}$$

Refer Fig. 5.4. Positive inventory during the cycle exists for time (Q – J)/D year in the cycle. Inventory is varying from (Q – J) to zero, and the average positive inventory is (Q – J)/2, which exists for a fraction of cycle time,

$$= \frac{(Q-J)/D}{Q/D}$$

$$= \frac{(Q-J)}{Q}$$

Annual inventory holding cost,

$$= \frac{(Q-J)}{2} \cdot \frac{(Q-J)}{Q} \cdot I$$

$$= \frac{(Q-J)^2 . I}{2Q} \qquad ...(5.35)$$

Similarly shortage quantity varies from zero to J, and the average shortages are (J/2), which exists for a fraction of cycle time,

$$= \frac{J/D}{Q/D}$$

$$= J/Q$$

Annual shortage (or backordering) cost,

$$= \frac{J}{2} \cdot \frac{J}{Q} \cdot K$$

$$= \frac{KJ^2}{2Q} \qquad ...(5.36)$$

As the annual demand is D and quantity Q is ordered frequently, number of orders placed in one year,

$$= \frac{D}{Q}$$

$$\text{Annual ordering cost} = \frac{D}{Q} . C \qquad ..(5.37)$$

Total annual cost consists of ordering, inventory holding and backordering cost. Adding equations (5.35), (5.36) and (5.37), total annual cost,

$$E = \frac{(Q-J)^2 I}{2Q} + \frac{KJ^2}{2Q} + \frac{DC}{Q} \qquad ...(5.38)$$

Now the objective is to minimize equation (5.38), and to obtain the optimum values of Q, J, and finally E. The problem can be solved by using any of the suitable methods discussed in the present book.

A wide variety of the situations can be modeled, some of which are as follows :

(*i*) Fractional backordering or partial backordering, *i.e.,* a situation in which a fraction of shortage quantity is not backordered.

(*ii*) In stead of instantaneous procurement, the replenishment rate is finite. This is also appropriate for production/manufacturing scenario. Shortages may or may not be included.

(*iii*) Quality defects can be incorporated in the production of an item and accordingly total relevant cost is formulated.

(*iv*) Multiple items are produced in a family production environment. Depending on the management decision, either common cycle time or different cycle time approach may be implemented. Constraint on the production time is imposed on the objective function.

(*v*) In the multi-item production environment, shelf life constraint may be incorporated. Storage time for an item must be less than or equal to the shelf life (if any) of that item. In order to deal with the problem, various policies are analyzed and optimum parameters are evaluated.

REFERENCES

1. G.S. Gupta. Managerial Economics. Tata McGraw-Hill, New Delhi, 1990.
2. Hamdy A. Taha. Operations Research. Pearson Education, Delhi, 2002.
3. Harvey M. Wagner. Principles of Operations Research. Prentice-Hall, 1975.
4. Mik Wisniewski and Tony Dacre. Mathematical Programming. McGraw-Hill Book Company (UK) Limited, 1990.
5. Sanjay Sharma. Mathematical Modeling of Production-Inventory Systems. Asian Books, New Delhi, 2004.
6. Singiresu S. Rao. Engineering Optimization. New Age International (P) Ltd., New Delhi, 1996.
7. Wayne L. Winston. Operations Research. Thomson Asia Pte. Ltd., 2003.